Lecture 1: Introduction

Michael Dinitz

August 26, 2025
601.433/633 Introduction to Algorithms

Welcome!

Introduction to (the theory of) algorithms

- How to design algorithms
- How to analyze algorithms

Prerequisites: Data Structures and MFCS/Discrete Math

- Small amount of review next lecture, but should be comfortable with asymptotic notation, basic data structures, basic combinatorics and graph theory.
- Undergrads from prereqs.
- "Informal" prerequisite: *mathematical maturity*

Instructors: Michael Dinitz, Jessica Sorrell

- First time with two instructors!

About me

- 10th time teaching this class (Fall 2014 - Fall 2025).
  - I'm still learning – let me know if you have suggestions!
  - Fall 2022: Sabbatical at Google Research-New York
  - Fall 2023: Parental leave

- Research in theoretical CS, particularly algorithms: approximation algorithms, graph algorithms, distributed algorithms, online algorithms.
- Also other parts of math (graph theory) and CS theory (algorithmic game theory, complexity theory) and theory of networking.

- Office hours: TBD.

Administrative Stuff

- TA: Nate Robinson and Yan Zhong (CS PhD students). Office hours TBD
- CAs: Many, still finalizing.
- Website: `https://introalgorithmsfall25.cs.jhu.edu`
  - Syllabus, schedule, lecture notes, office hours, . . .
  - Courselore for discussion/announcements
  - Gradescope for homeworks/exams.
- Textbook: CLRS (fourth edition)

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- Textbook: CLRS (fourth edition)
- Class a bit different than in the past!
  - Fewer homeworks, in-class quizzes, “recitation-like” office hours

Assignments

Homeworks:

- Approximately every 2 weeks, posted on website (HW1 out next Tuesday)
- *Must* be typeset (LaTeX preferred, not required)
- Work in groups of ≤ 3 (highly recommended). But *individual* writeups.
  - Work together at a whiteboard to solve, then write up yourself.
  - Write group members at top of homework
- 120 late hours (5 late days) total

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- Once/week
- Should be short and simple: did you understand the last few lectures/homeworks?

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Exams: Midterm, final.

- Midterm: In-class (75 minutes), traditional, closed book
- Final: in person, scheduled by registrar. 3 hours, traditional, closed book.

Grading Breakdown

Grading: 30% homework, 10% quizzes, 20% midterm, 40% final exam,

- "Curve": Historically, average $\approx$ B+. About 50% A's, 50% B's, a few below.
  - Curve only helps! Someone else doing well does not hurt you.
  - Be collaborative and helpful (within guidelines).

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  - Using ChatGPT or other LLMs.
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  - Use the internet, classmates, other resources to understand concepts from class, not to help with assignments.
- In previous years, punishments have included zero on assignment, grade penalty, mark on transcript, etc. $\geq \mathbf{1}$ person has had PhD acceptance revoked.

LLMs

LLMs can be very useful tools!

- If you overuse them, you will not actually learn the material
- If you're no better than an LLM, why should anyone hire you or care what you have to say?

Often incorrect, and will double down on mistakes

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- **o3**: No online algorithm can do this – there is an information-theoretic lower bound.
- **Me**: Great! Can you show me the proof of this?
- **o3**: complicated lower bound argument

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- **o3**: "You are right – the example I gave . . . cannot be used to prove a lower bound . . . Below I sketch how one can repair the lower-bound argument"
- **Me**: I still don't understand this part of the argument. Can you give it to me more formally?
- **o3**: long complicated argument

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- **o3**: "You are right . . . Thank you for catching the mistake. . . . I apologize for the earlier confusion. . . . I will need to rethink the question from the beginning"
- **Me**: "OK, please do so"

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- **Me**: "I don't understand how you get from (8) to (9)"
- **o3**: "You are completely right . . . My previous message was therefore wrong once again; the step from (8) to (9) is unjustified. Thank you for pointing this out. . . . I apologise for the repeated confusion that my earlier, faulty arguments have caused."

On Learning

- Learning is challenging
- The challenge is typically proportional to *how much you're learning*
- You are rewiring your brain to be able to think in new ways! In this class in particular, you are practicing convergent creativity, thinking precisely, rigorously, adversarially, recursively, etc. These are broadly useful skills!
- No amount of Professor Sorrell and I talking at you can give you these skills. You can think of the course staff like personal trainers. We can recommend exercises, demonstrate technique, but we can't do them for you. At the end of the day, if you want your brain to adapt to the challenges of this course, you need to practice (the more consistently, the better!)
- Practice should be challenging, but not painful! If you're struggling, please talk us. We're here to help you succeed!

Course Overview

- Introduction to *Theory* of Algorithms: math not programming.
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  - Running time: worst-case, average-case, worst-case expected, amortized, ...
  - Space usage
  - and more!

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  - and more!
- This class: mostly correctness and asymptotic running time, focus on worst-case

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- Obviously want to prove correctness!
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- We will focus on how algorithm "scales": how running times change as input grows. Hard to determine experimentally.
- Most importantly: want to *understand*.
  - Experiments can (maybe) convince you that something is true. But can't tell you why!

Example 1: Multiplication

Multiplication I

Often an obvious way to solve a problem just from the definition. But might not be the right way!

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Running time:

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- Could be $\Theta(2^n)$. *Exponential* in size of input ($2n$).

Multiplication II

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```
          110110  = 54
x         101001  = 41
----------------
          110110
       110110
+    110110
----------------
    100010100110  = 2 + 4 + 32 + 128 + 2048 = 2214
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Running time:

- $O(n)$ column additions, each takes $O(n)$ time $\implies$ $O(n^2)$ time.
- Better than obvious algorithm!

Multiplication III

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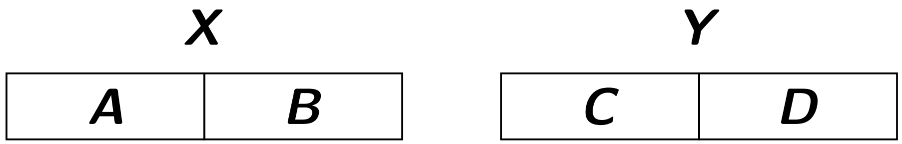
$X = 2^{n/2}A + B$

$Y = 2^{n/2}C + D$



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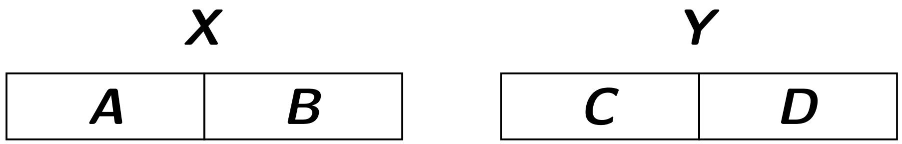

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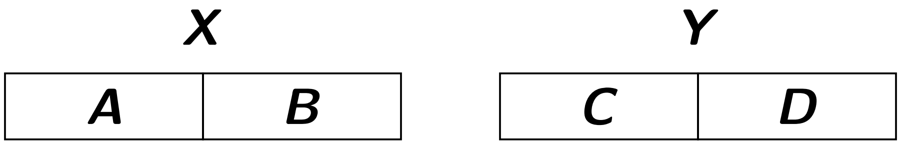

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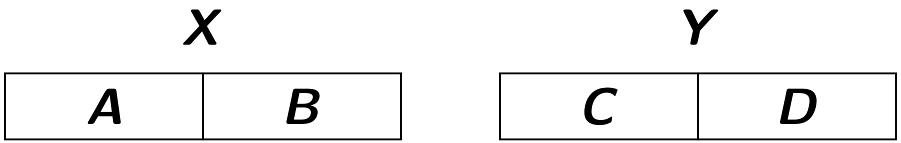

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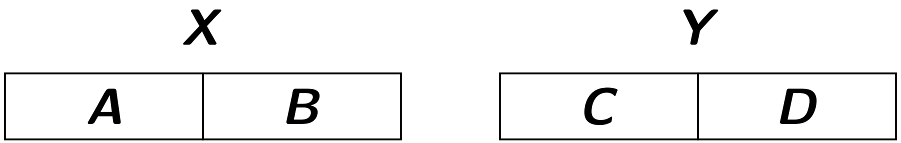

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Running Time: $T(n) = 4T(n/2) + cn$

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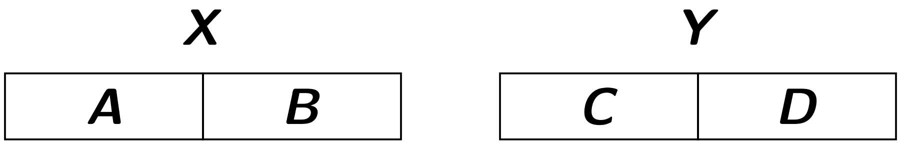

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Running Time: $T(n) = 4T(n/2) + cn \implies T(n) = O(n^2)$

Karatsuba Multiplication

Rewrite equation for XY:

$$\begin{aligned} XY &= 2^n AC + 2^{n/2} AD + 2^{n/2} BC + BD \\ &= 2^{n/2}(A+B)(C+D) + (2^n - 2^{n/2})AC + (1 - 2^{n/2})BD \end{aligned}$$

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$\implies T(n) = O(n^{\log_2 3}) \approx O(n^{1.585})$

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Theorem (Karp)

There is an $O(n \log^2 n)$*-time algorithm for multiplication.*

Uses *Fast Fourier Transform* (FFT)

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Theorem (Harvey and van der Hoeven '19)

There is an $O(n \log n)$-time algorithm for multiplication.

Example 2: Matrix Multiplication

Matrix Multiplication: Definition

Given $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n \times n}$, compute $\boldsymbol{XY} \in \mathbb{R}^{n \times n}$

- $(\boldsymbol{XY})_{ij} = \sum_{k=1}^{n} \boldsymbol{X}_{ik} \boldsymbol{Y}_{kj}$
- Don't worry for now about representing real numbers
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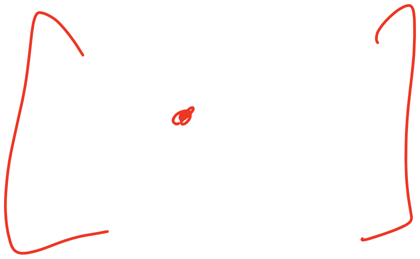

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Running time:

- $O(n^2)$ entries, each entry takes n multiplications and $n-1$ additions $\implies O(n^3)$ time.

Strassen I

Break $\boldsymbol{X}$ and $\boldsymbol{Y}$ each into four $(\boldsymbol{n}/2) \times (\boldsymbol{n}/2)$ matrices:

$$\boldsymbol{X} = \begin{array}{|c|c|} \hline \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \\ \hline \end{array} \qquad \boldsymbol{Y} = \begin{array}{|c|c|} \hline \boldsymbol{E} & \boldsymbol{F} \\ \hline \boldsymbol{G} & \boldsymbol{H} \\ \hline \end{array}$$

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Recursive algorithm: compute eight $(\boldsymbol{n}/2) \times (\boldsymbol{n}/2)$ matrix multiplies, four additions

Strassen II

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Improve on this?

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Running time: $T(n) = 7T(n/2) + c'n^2 \implies T(n) = O(n^{\log_2 7}) \approx O(n^{2.8074})$.

Further Progress

- Coppersmith and Winograd '90: $O(n^{2.375477})$
- Virginia Vassilevska Williams '13: $O(n^{2.3728642})$
- François Le Gall '14: $O(n^{2.3728639})$
- Josh Alman and Virginia Vassilevska Williams '21: $O(n^{2.3728596})$
- Virginia Vassilevska Williams, Yinzhan Xu, Zixuan Xu, and Renfei Zhou '24: $O(n^{2.371552})$.
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Is there an algorithm for matrix multiplication in $O(n^2)$ time?

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If you answer this (with proof!), automatic A+ in course and PhD

See you Thursday!