Lecture 1: Introduction

Jessica Sorrell

August 26, 2025
601.433/633 Introduction to Algorithms
(Slides by Michael Dinitz, tweaked for our section)

Welcome!

Introduction to (the theory of) algorithms

- How to design algorithms
- How to analyze algorithms

Prerequisites: Data Structures and MFCS/Discrete Math

- Small amount of review next lecture, but should be comfortable with asymptotic notation, basic data structures, basic combinatorics and graph theory.
- Undergrads from prereqs.
- "Informal" prerequisite: *mathematical maturity*

Instructors: Michael Dinitz, Jessica Sorrell

- First time with two instructors!

About me

- 1st time teaching this class!
  - I have a lot to learn – let me know if you have suggestions!
  - Started at Hopkins last fall
  - Previously: postdoc at Penn, PhD at University of California, San Diego

- Research in theoretical CS, specifically machine learning theory: interested in understanding the limits of learning algorithms and how to make the models they output more reliable/trustworthy
- Tools from theoretical computer science I use in my research: probability theory, cryptography, algorithm design and analysis, NP-hardness

- Office hours: TBD.

Administrative Stuff

- TA: Nate Robinson and Yan Zhong (CS PhD students). Office hours TBD
- CAs: Many, still finalizing.
- Website: `https://introalgorithmsfall25.cs.jhu.edu`
  - Syllabus, schedule, lecture notes, office hours, ...
  - Courselore for discussion/announcements
  - Gradescope for homeworks/exams.
- Textbook: CLRS (fourth edition)

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- Class a bit different than in the past!
  - Fewer homeworks, in-class quizzes, "recitation-like" office hours

Assignments

Homeworks:

- Approximately every 2 weeks, posted on website (HW1 out next Tuesday)
- *Must* be typeset (LaTeX preferred, not required)
- Work in groups of ≤ 3 (highly recommended). But *individual* writeups.
  - Work together at a whiteboard to solve, then write up yourself.
  - Write group members at top of homework
- 120 late hours (5 late days) total

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- In-class, approx. 10-15 minutes
- Once/week
- Should be short and simple: did you understand the last few lectures/homeworks?

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Exams: Midterm, final.

- Midterm: In-class (75 minutes), traditional, closed book
- Final: in person, scheduled by registrar. 3 hours, traditional, closed book.

Grading Breakdown

Grading: 30% homework, 10% quizzes, 20% midterm, 40% final exam,

- "Curve": Historically, average $\approx$ B+. About 50% A's, 50% B's, a few below.
  - Curve only helps! Someone else doing well does not hurt you.
  - Be collaborative and helpful (within guidelines).

On Learning

- Learning is challenging.
- Your brain is adapting to new environmental stimuli by reorganizing and forming new connections, but it would really rather conserve energy and stick with what feels familiar
- In this class in particular, you are practicing convergent creativity, thinking precisely, rigorously, adversarially, recursively, etc. These are broadly useful skills!
- No amount of Professor Dinitz and I talking at you can give you these skills. You can think of the course staff like personal trainers. We can recommend exercises, demonstrate technique, but we can't do them for you. At the end of the day, if you want your brain to adapt to the challenges of this course, you need to practice (the more consistently, the better!)
- Practice should be challenging, but not painful! If you're struggling, please talk us. We're here to help you succeed.

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  - Using ChatGPT or other LLMs.
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- Just solve the problems with your group and write them up yourself!
  - Use the internet, classmates, other resources to understand concepts from class, not to help with assignments.
- In previous years, punishments have included zero on assignment, grade penalty, mark on transcript, etc. $\geq \mathbf{1}$ person has had PhD acceptance revoked.

LLMs

LLMs can be very useful tools!

- If you overuse them, you will not actually learn the material
- If you're no better than an LLM, why should anyone hire you or care what you have to say?

Often incorrect, and will double down on mistakes

LLMs

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LLMs

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- **o3**: No online algorithm can do this – there is an information-theoretic lower bound.
- **Me**: Great! Can you show me the proof of this?
- **o3**: complicated lower bound argument

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- **o3**: "You are right – the example I gave . . . cannot be used to prove a lower bound . . . Below I sketch how one can repair the lower-bound argument"
- **Me**: I still don't understand this part of the argument. Can you give it to me more formally?
- **o3**: long complicated argument

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- **o3**: "You are right ... Thank you for catching the mistake. ... I apologize for the earlier confusion. ... I will need to rethink the question from the beginning"
- **Me**: "OK, please do so"

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- **o3**: "Below is a completely explicit construction – it uses nothing more than the one–dimensional line"
- **Me**: "I don't understand how you get from (8) to (9)"
- **o3**: "You are completely right . . . My previous message was therefore wrong once again; the step from (8) to (9) is unjustified. Thank you for pointing this out. . . . I apologise for the repeated confusion that my earlier, faulty arguments have caused."

Course Overview

- Introduction to *Theory* of Algorithms: math not programming.
- Two goals: how to *design* algorithms, and how to *analyze* algorithms.
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  - Correctness: it does solve the problem.
  - Running time: worst-case, average-case, worst-case expected, amortized, ...
  - Space usage
  - and more!

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  - and more!
- This class: mostly correctness and asymptotic running time, focus on worst-case

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- Obviously want to prove correctness!
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- We will focus on how algorithm "scales": how running times change as input grows. Hard to determine experimentally.
- Most importantly: want to *understand*.
  - Experiments can (maybe) convince you that something is true. But can't tell you why!

Example 1: Multiplication

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- Add X to itself Y times: $X + X + \cdots + X$. Or add Y to itself X times: $Y + Y + \cdots + Y$.

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- $\Theta(Y)$ or $\Theta(X)$ (assuming constant-time adds).
- Could be $\Theta(2^n)$. *Exponential* in size of input ($2n$).

Multiplication II

Better idea?

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```
          110110   = 54
x         101001   = 41
----------------
          110110
       110110
+    110110
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    100010100110   = 2 + 4 + 32 + 128 + 2048 = 2214
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```

Running time:

- $O(n)$ column additions, each takes $O(n)$ time $\implies O(n^2)$ time.
- Better than obvious algorithm!

Multiplication III

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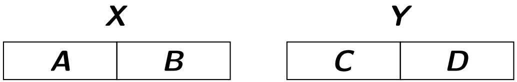

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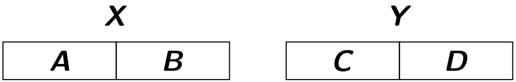

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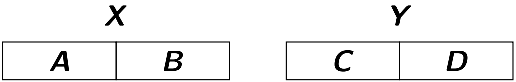

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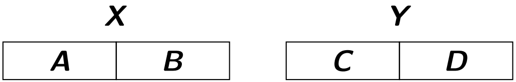

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Four $n/2$-bit multiplications, three shifts, three $O(n)$-bit adds.

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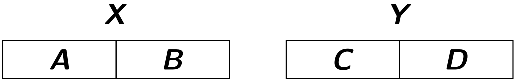

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Running Time: $T(n) = 4T(n/2) + cn$

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Four $n/2$-bit multiplications, three shifts, three $O(n)$-bit adds.

Running Time: $T(n) = 4T(n/2) + cn \implies T(n) = O(n^2)$

Karatsuba Multiplication

Rewrite equation for $\boldsymbol{XY}$:

$$\begin{aligned} \boldsymbol{XY} &= 2^{\boldsymbol{n}}\boldsymbol{AC} + 2^{\boldsymbol{n}/2}\boldsymbol{AD} + 2^{\boldsymbol{n}/2}\boldsymbol{BC} + \boldsymbol{BD} \\ &= 2^{\boldsymbol{n}/2}(\boldsymbol{A} + \boldsymbol{B})(\boldsymbol{C} + \boldsymbol{D}) + (2^{\boldsymbol{n}} - 2^{\boldsymbol{n}/2})\boldsymbol{AC} + (1 - 2^{\boldsymbol{n}/2})\boldsymbol{BD} \end{aligned}$$

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Three $\boldsymbol{n}/2$-bit multiplications, $\boldsymbol{O}(1)$ shifts and $\boldsymbol{O}(\boldsymbol{n})$-bit adds.

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$\implies T(n) = O(n^{\log_2 3}) \approx O(n^{1.585})$

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Theorem (Karp)

There is an $O(n \log^2 n)$*-time algorithm for multiplication.*

Uses *Fast Fourier Transform* (FFT)

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Theorem (Harvey and van der Hoeven '19)

There is an $\boldsymbol{O(n \log n)}$*-time algorithm for multiplication.*

Example 2: Matrix Multiplication

Matrix Multiplication: Definition

Given $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{\boldsymbol{n} \times \boldsymbol{n}}$, compute $\boldsymbol{XY} \in \mathbb{R}^{\boldsymbol{n} \times \boldsymbol{n}}$

- $(\boldsymbol{XY})_{\boldsymbol{ij}} = \sum_{\boldsymbol{k}=1}^{\boldsymbol{n}} \boldsymbol{X}_{\boldsymbol{ik}} \boldsymbol{Y}_{\boldsymbol{kj}}$
- Don't worry for now about representing real numbers
- Assume multiplication in $\boldsymbol{O}(1)$ time

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Algorithm from definition:

- For each $i, j \in \{1, 2, \ldots, n\}$, compute $(\boldsymbol{XY})_{ij}$ using formula.

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Algorithm from definition:

- For each $i, j \in \{1, 2, \ldots, n\}$, compute $(XY)_{ij}$ using formula.

Running time:

- $O(n^2)$ entries, each entry takes n multiplications and $n - 1$ additions $\implies O(n^3)$ time.

Strassen I

Break $\boldsymbol{X}$ and $\boldsymbol{Y}$ each into four $(\boldsymbol{n}/2) \times (\boldsymbol{n}/2)$ matrices:

$$\boldsymbol{X} = \begin{array}{|c|c|} \hline \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \\ \hline \end{array} \qquad \boldsymbol{Y} = \begin{array}{|c|c|} \hline \boldsymbol{E} & \boldsymbol{F} \\ \hline \boldsymbol{G} & \boldsymbol{H} \\ \hline \end{array}$$

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So can rewrite $\boldsymbol{XY}$:

$$\boldsymbol{XY} = \begin{bmatrix} \boldsymbol{AE} + \boldsymbol{BG} & \boldsymbol{AF} + \boldsymbol{BH} \\ \boldsymbol{CE} + \boldsymbol{DG} & \boldsymbol{CF} + \boldsymbol{DH} \end{bmatrix}$$

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Recursive algorithm: compute eight $(\boldsymbol{n}/2) \times (\boldsymbol{n}/2)$ matrix multiplies, four additions

Strassen II

$$\boldsymbol{XY} = \begin{array}{|c|c|} \hline \boldsymbol{AE} + \boldsymbol{BG} & \boldsymbol{AF} + \boldsymbol{BH} \\ \hline \boldsymbol{CE} + \boldsymbol{DG} & \boldsymbol{CF} + \boldsymbol{DH} \\ \hline \end{array}$$

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Running time: $T(n) = 8T(n/2) + cn^2 \implies T(n) = O(n^3)$.

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Improve on this?

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Strassen IV

$$M_1 = (A + D)(E + H) \qquad M_2 = (C + D)E \qquad M_3 = A(F - H)$$
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Only *seven* $(n/2) \times (n/2)$ matrix multiplies, $O(1)$ additions

Running time: $T(n) = 7T(n/2) + c'n^2 \implies T(n) = O(n^{\log_2 7}) \approx O(n^{2.8074})$.

Further Progress

- Coppersmith and Winograd '90: $O(n^{2.375477})$
- Virginia Vassilevska Williams '13: $O(n^{2.3728642})$
- François Le Gall '14: $O(n^{2.3728639})$
- Josh Alman and Virginia Vassilevska Williams '21: $O(n^{2.3728596})$
- Virginia Vassilevska Williams, Yinzhan Xu, Zixuan Xu, and Renfei Zhou '24: $O(n^{2.371552})$.
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Is there an algorithm for matrix multiplication in $O(n^2)$ time?

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> Is there an algorithm for matrix multiplication in $O(n^2)$ time?

If you answer this (with proof!), automatic A+ in course and PhD

See you Thursday!