# Lecture 9: Disjoint Sets / Union-Find

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September 25, 2025

601.433/633 Introduction to Algorithms

# Introduction

Informal: Universe of elements, want to maintain *disjoint sets*.

Slightly more formally:

- Make-Set($x$): create a new set containing just $x$ (i.e., $\{x\}$)
- Union($x, y$): Replace set containing $x$ ($S$) and set containing $y$ ($T$) with single set $S \cup T$
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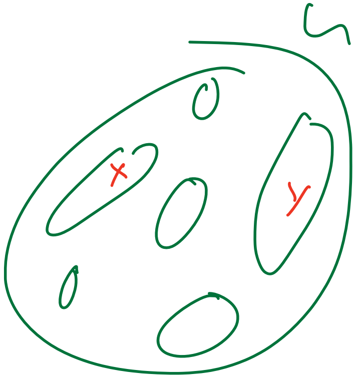


Rules: every set has a *unique* representative.

- If $x$ and $y$ are in same set, Find($x$) = Find($y$)
- If $x$ and $y$ are in different sets, then Find($x$) $\neq$ Find($y$)
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Note: disjoint (and partition) by construction!

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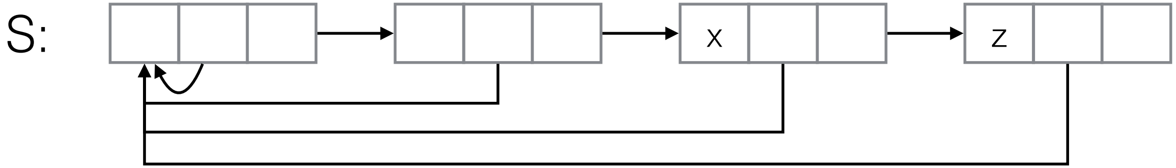
Notation and Notes:

- $m$ operations total
- $n$ of which are Make-Sets (so $n$ elements)
- Assume have pointer/access to elements we care about (like last class)

# First Approach: Lists

Linked list for each set.

- Representative of set is head (first element on list)
- Each element has pointer to head and to next element, so stored as triple: (element, head, next)



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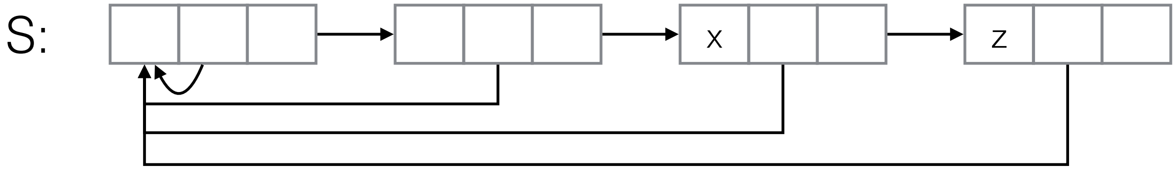
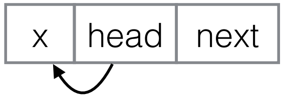



Make-Set($x$):



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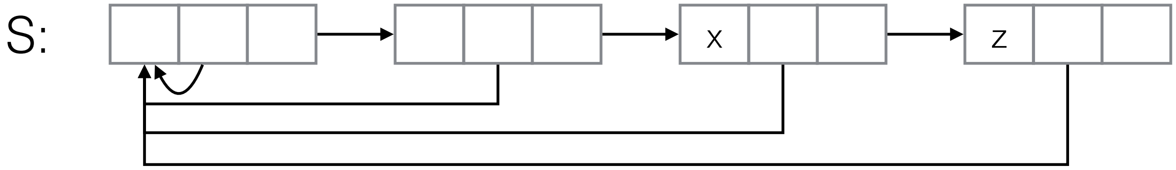
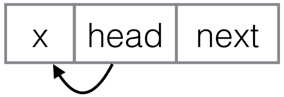
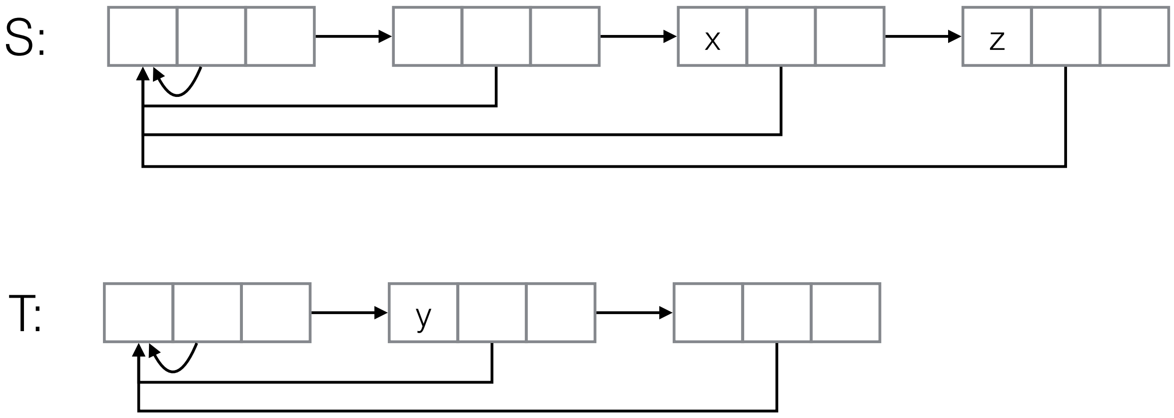



Make-Set($\boldsymbol{x}$):

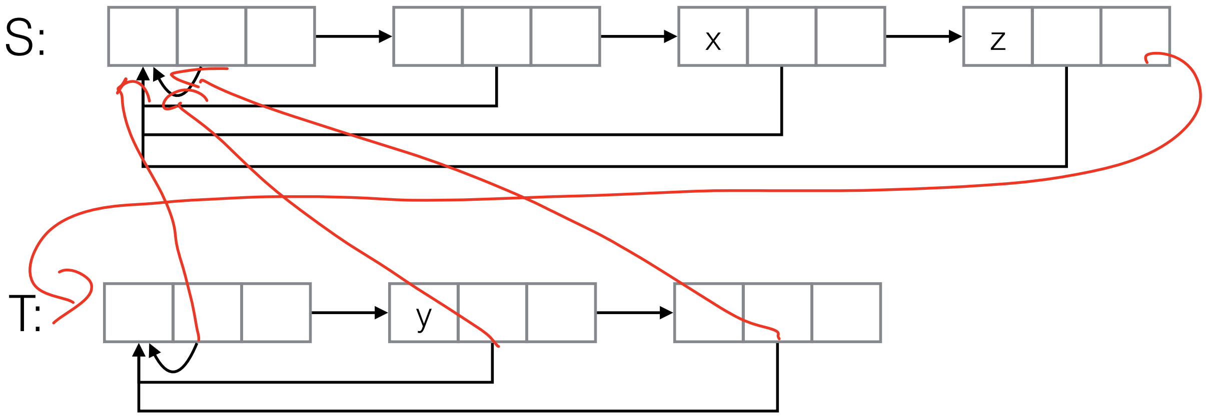



Find($\boldsymbol{x}$): return $\boldsymbol{x} \rightarrow$ head

# Union($x, y$)



# Union($\boldsymbol{x}, \boldsymbol{y}$)




Obvious approach:

- Walk down $\boldsymbol{S}$ to final element $\boldsymbol{z}$ (starting from $\boldsymbol{x}$)
- Set $\boldsymbol{z} \rightarrow \text{next} = \boldsymbol{y} \rightarrow \text{head}$
- Walk down $\boldsymbol{T}$, set every elements head pointer to $\boldsymbol{x} \rightarrow \text{head}$

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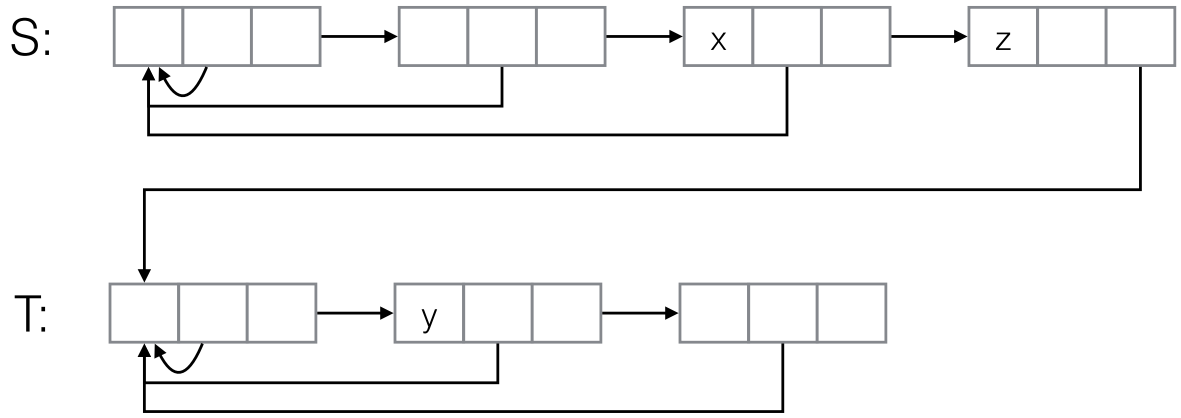

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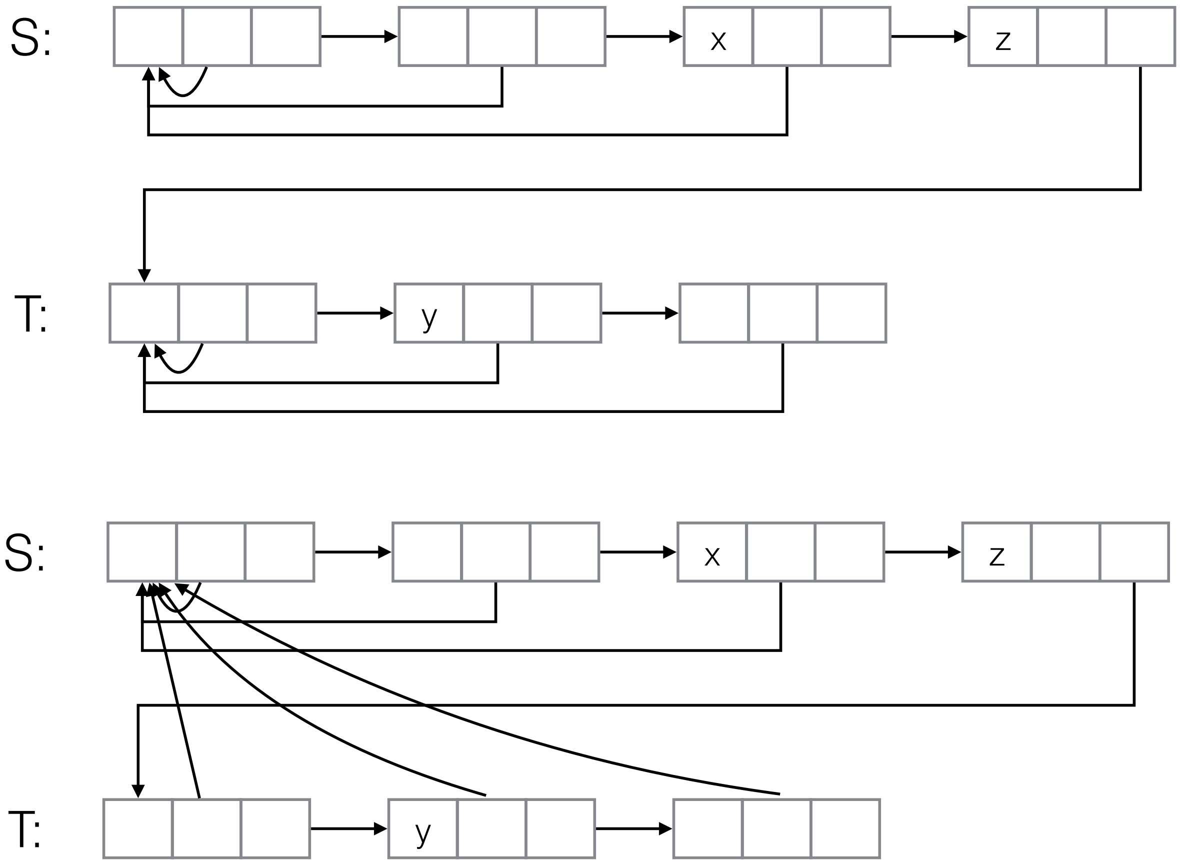

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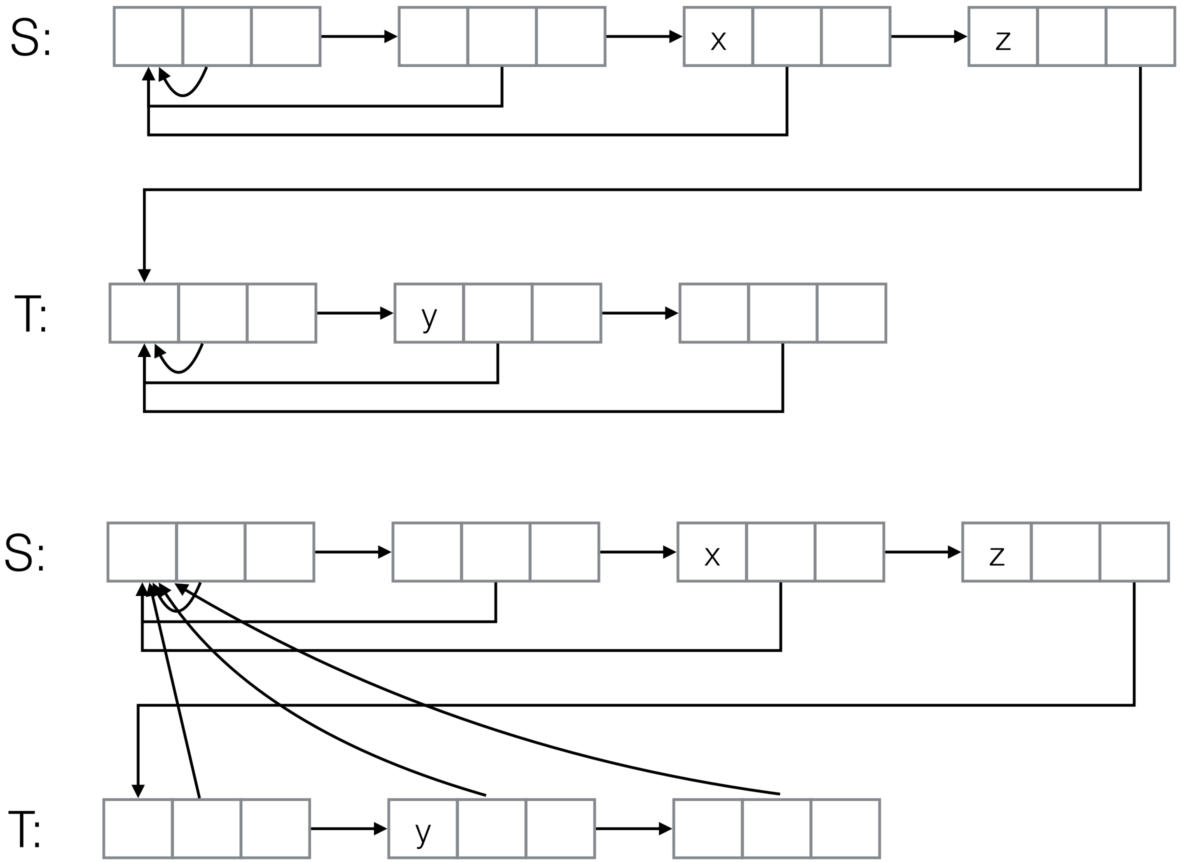


Running time:

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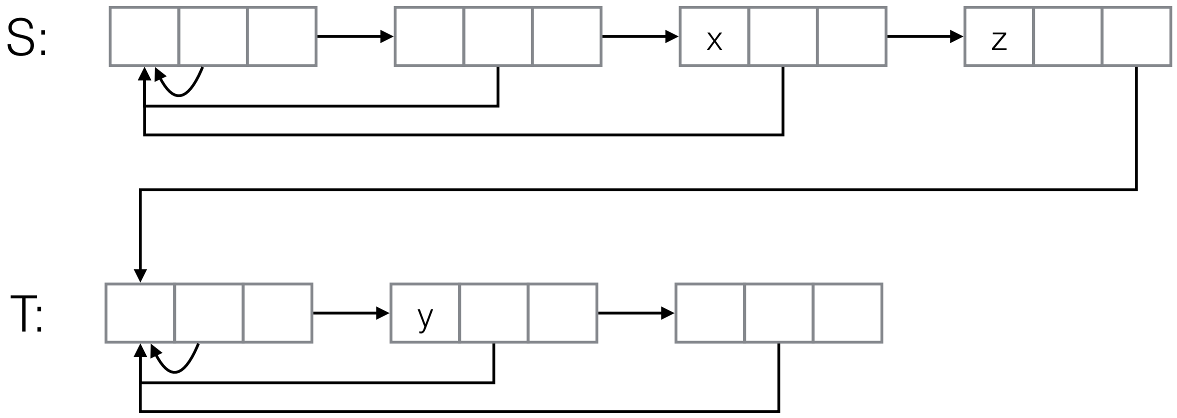
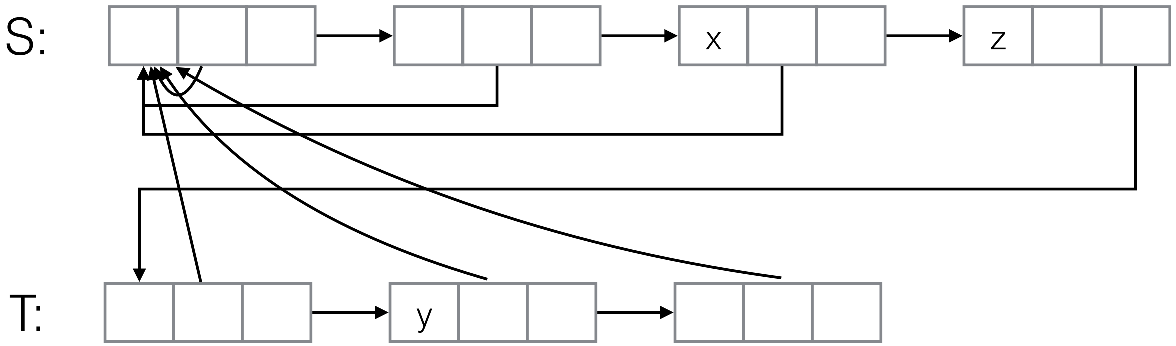



Running time: $O(|S| + |T|)$



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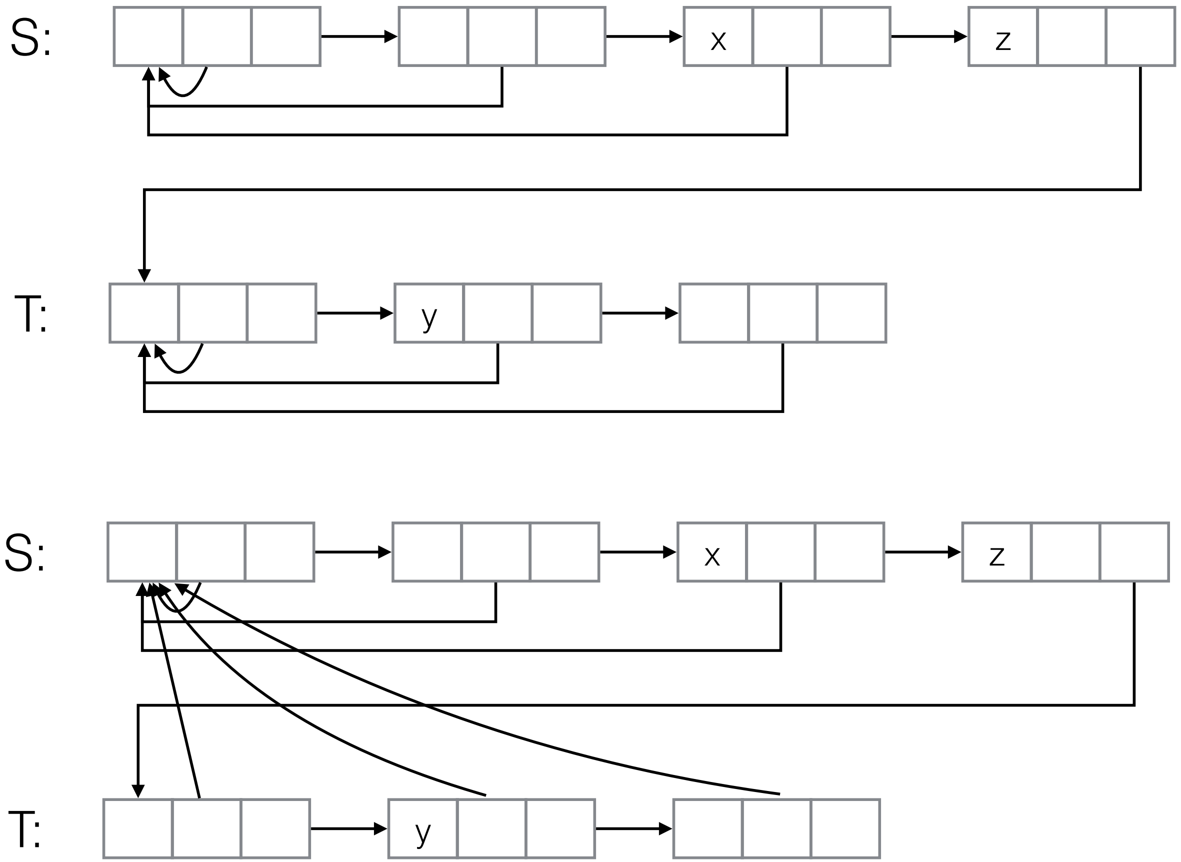


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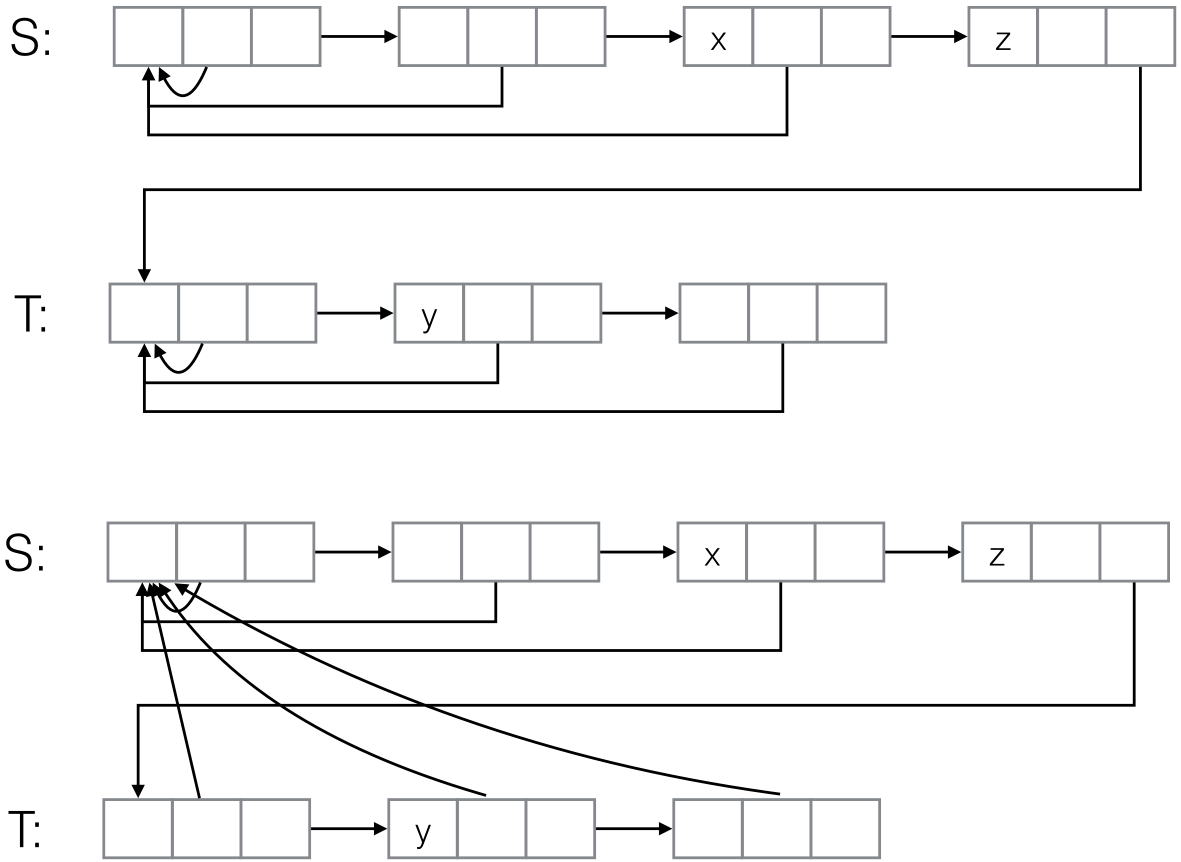


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Since $|S|, |T|$ could be $\Theta(n)$, can only say $O(n)$ for Unions

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Observation: don't need to preserve ordering inside the Union!

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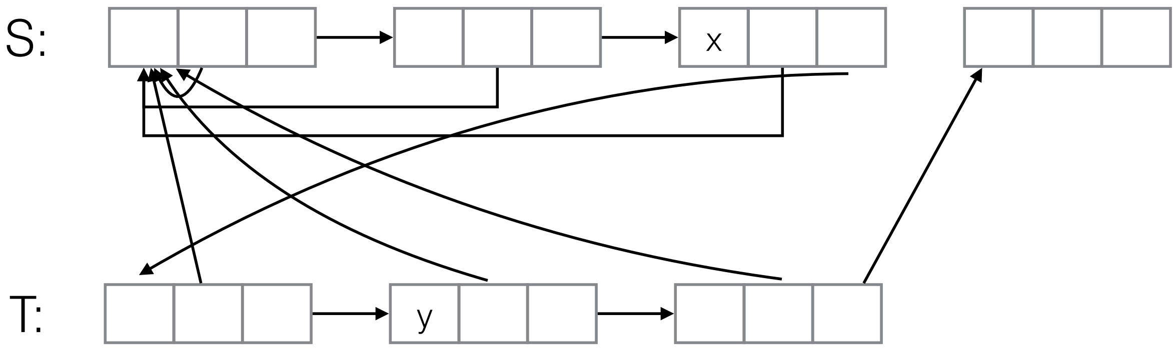

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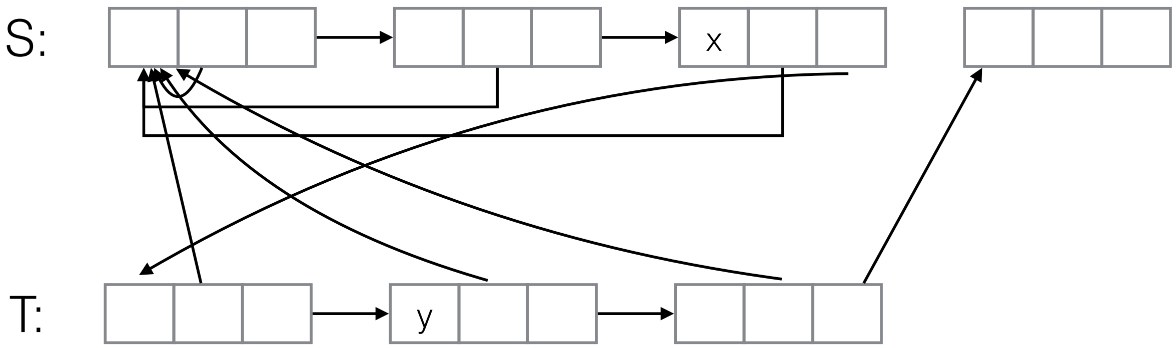


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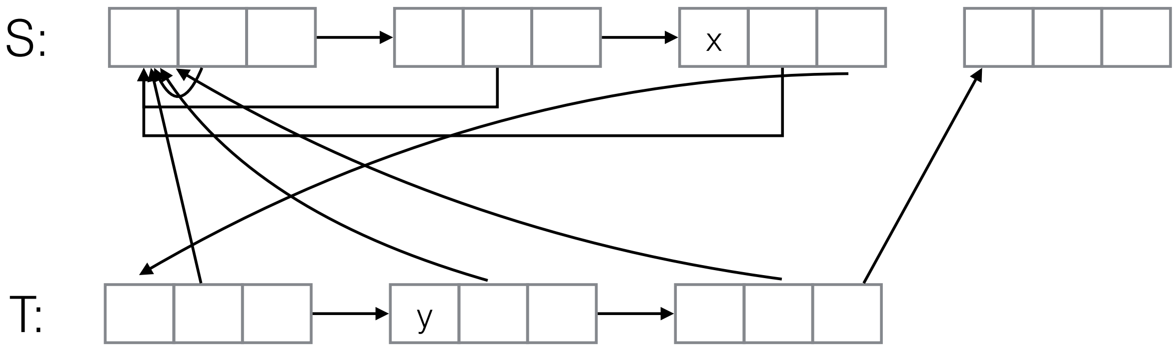


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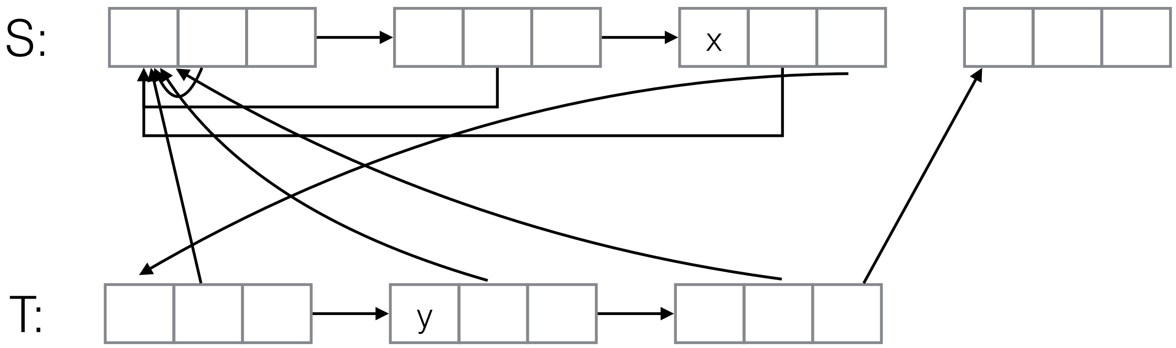


Running time: $O(|T|)$

- Still can't say anything better than $O(n)$

# Even more improved Union($\boldsymbol{x}, \boldsymbol{y}$)

Observation: Why splice $\boldsymbol{T}$ into $\boldsymbol{S}$? Could also splice $\boldsymbol{S}$ into $\boldsymbol{T}$.

- Time $\boldsymbol{O}(|\boldsymbol{S}|)$

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Splice smaller into bigger!

- Store size of set in head node.
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**Theorem**

*The amortized cost of Find and Union is $O(1)$, and the amortized cost of Make-Set is $O(\log n)$.*

**Corollary**

*The total running time is $O(m + n \log n)$.*

# Amortized Analysis of List Algorithm

Banking/accounting argument: bank for every element

- When an element is created (via Make-Set), add $\log n$ tokens to its bank
- Find does not affect banks
- When doing Union, take token from bank of each element in smaller set.

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- Size of set containing $e$ at least doubles!
- Can only happen at most $\log n$ times. $\square$

# Amortized Analysis of List Algorithm (cont'd)

Make-Set:

- True cost: $O(1)$
- Change in banks: $\log n$

$\implies$ Amortized cost: $O(1) + O(\log n) = O(\log n)$

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Find:

- True cost: $O(1)$
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- Change in banks: $0$

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Union:

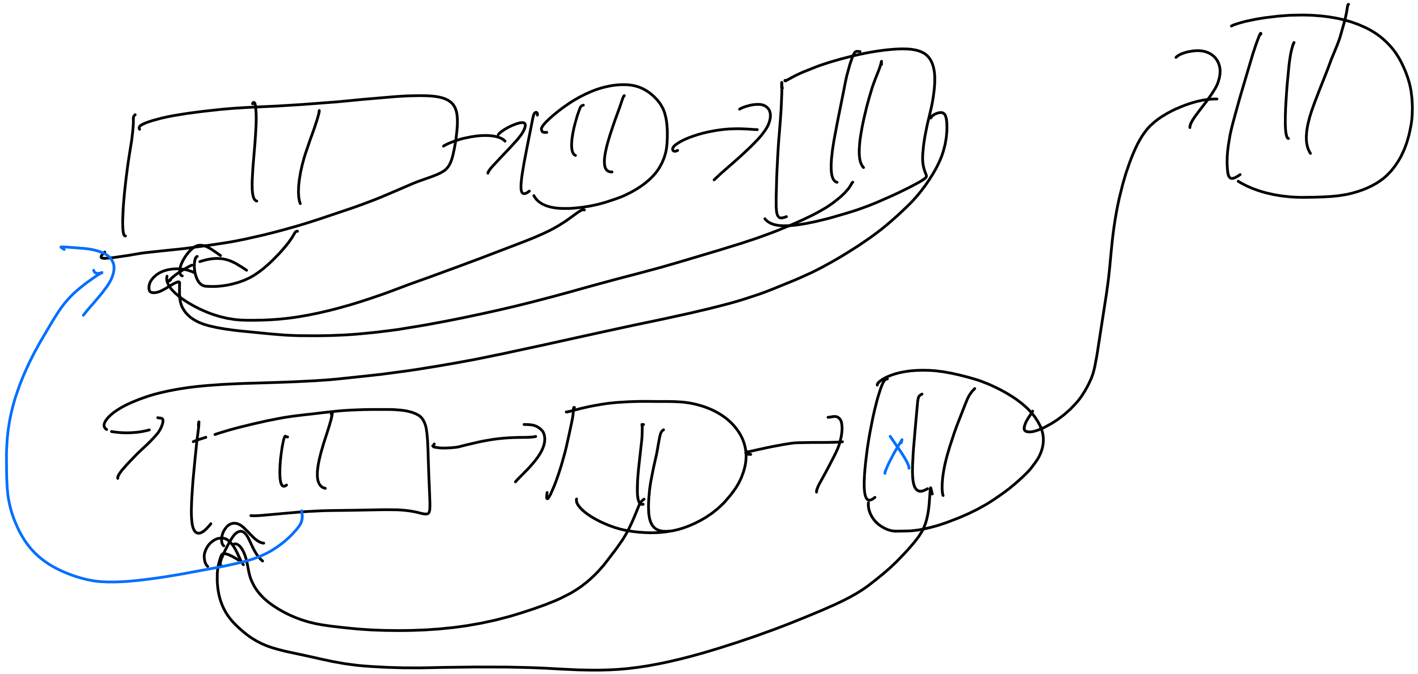
- True cost: $\min(|S|, |T|)$
- Change in banks: $-\min(|S|, |T|)$

$\Longrightarrow$ Amortized cost: $\min(|S|, |T|) - \min(|S|, |T|) = 0 = O(1)$.

# Even Better

Starting idea: want to make Unions faster, willing to make Finds a little slower.

- Slow part of Union: updating all head pointers in smaller list.
- Don't do it!



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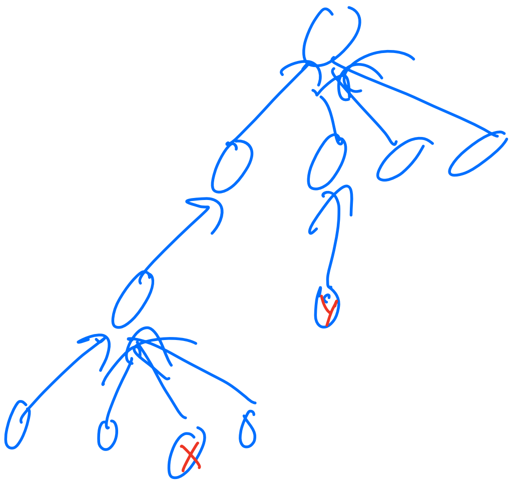


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Idea 2: *Union By Rank*

- Size of set was important for lists, less important for trees.
- Choose which set to splice into which by *rank* of trees (related to height)

# Main Result

**Theorem**

*When using Path Compression and Union By Rank, total time at most* $O(m \log^* n)$.

$\log^*$: iterated $\log_2$.

- $\log^* n = \#$ times apply $\log_2$ until get to $1$

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Stronger theorem: total time at most $O(m \cdot \alpha(m, n))$.

- $\alpha(m, n)$: inverse Ackermann function. Grows even slower than $\log^*$.
- See CLRS for details

# Formal Procedures: Make-Set and Find

Make-Set($\boldsymbol{x}$): Set $\boldsymbol{x} \rightarrow \boldsymbol{rank} = \mathbf{0}$ and $\boldsymbol{x} \rightarrow \boldsymbol{parent} = \boldsymbol{x}$

- Running time: $\boldsymbol{O(1)}$.

# Formal Procedures: Make-Set and Find

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Find($x$): Walk from $x$ to root, and return root. Set parent pointers of $x$ and all ancestors to root.

- If $x \to \boldsymbol{parent} = x$ then return $x$
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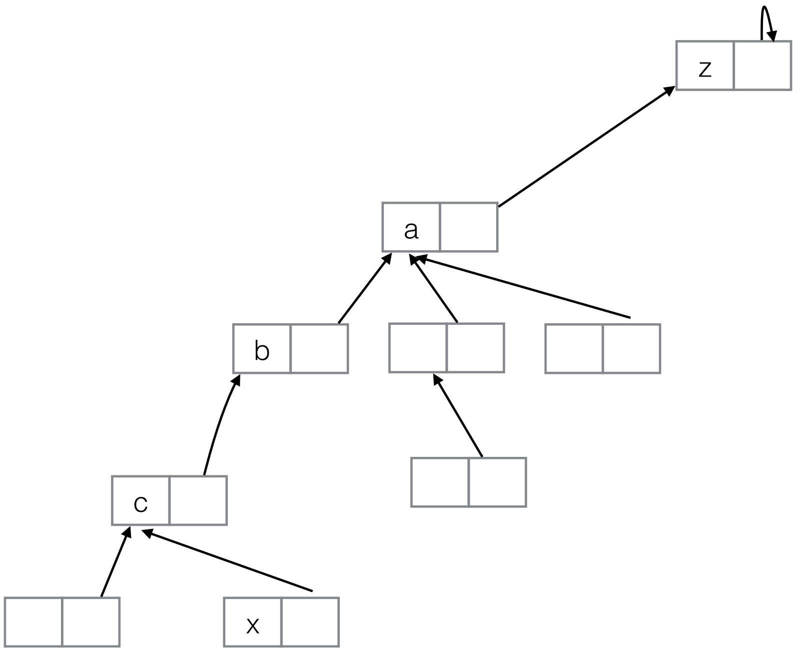
Running time of Find: depth of $x$ (distance to root)

# Find example



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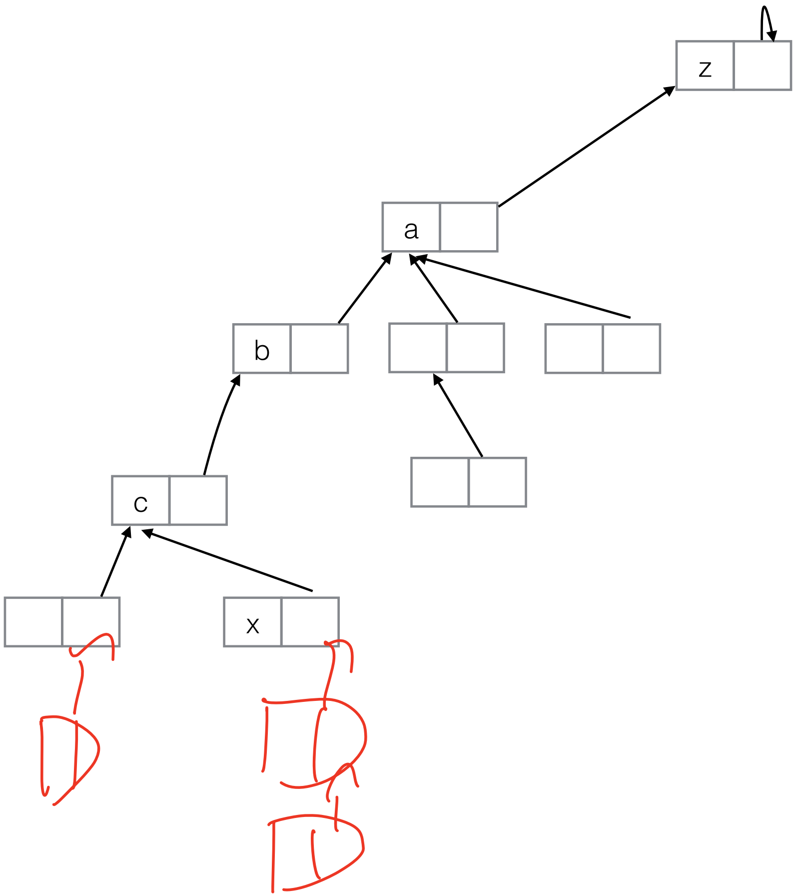
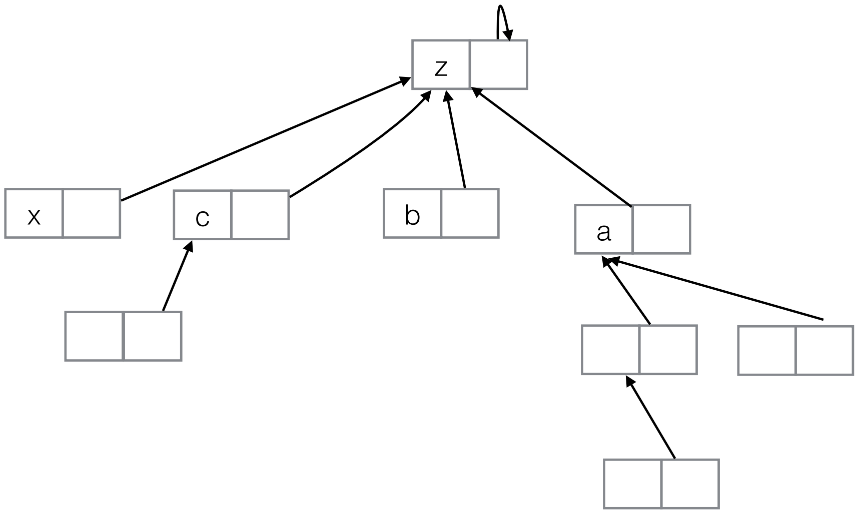
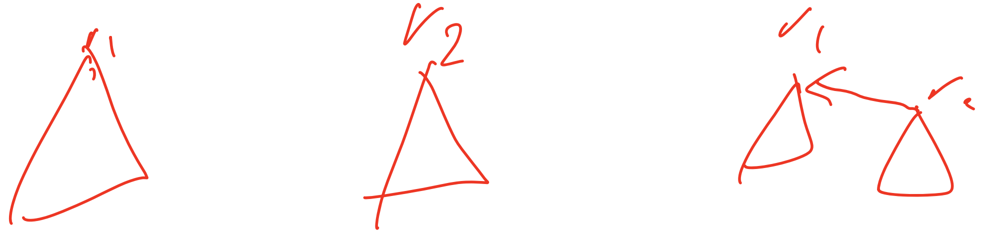


# Formal Procedure: Union

Link($\boldsymbol{r_1}, \boldsymbol{r_2}$): Only applied to root nodes

- If $\boldsymbol{r_1} \to \boldsymbol{rank} > \boldsymbol{r_2} \to \boldsymbol{rank}$, set $\boldsymbol{r_2} \to \boldsymbol{parent} = \boldsymbol{r_1}$
- If $\boldsymbol{r_2} \to \boldsymbol{rank} > \boldsymbol{r_1} \to \boldsymbol{rank}$, set $\boldsymbol{r_1} \to \boldsymbol{parent} = \boldsymbol{r_2}$
- If $\boldsymbol{r_1} \to \boldsymbol{rank} = \boldsymbol{r_2} \to \boldsymbol{rank}$, set $\boldsymbol{r_2} \to \boldsymbol{parent} = \boldsymbol{r_1}$ and increment $\boldsymbol{r_1} \to \boldsymbol{rank}$.



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Union($\boldsymbol{x}, \boldsymbol{y}$): Link(Find($\boldsymbol{x}$), Find($\boldsymbol{y}$))

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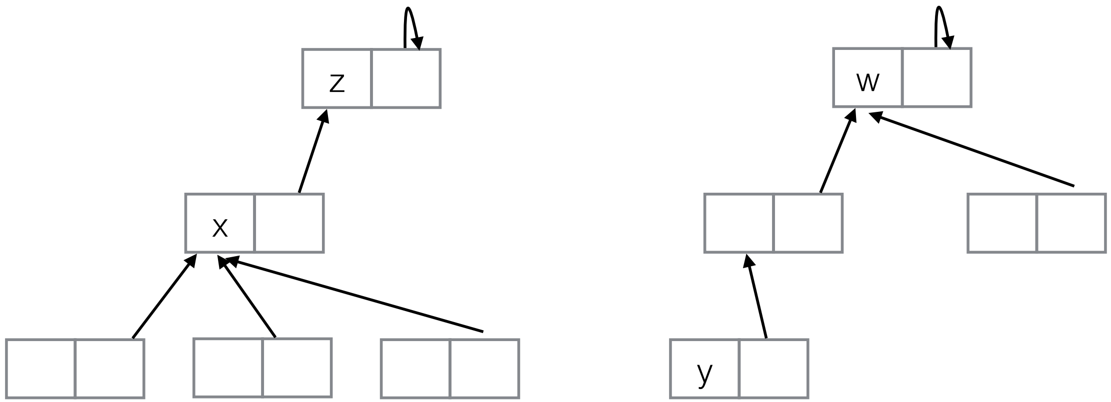
Union($x, y$): Link(Find($x$), Find($y$))

- Running time: depth($x$) + depth($y$)

# Union example



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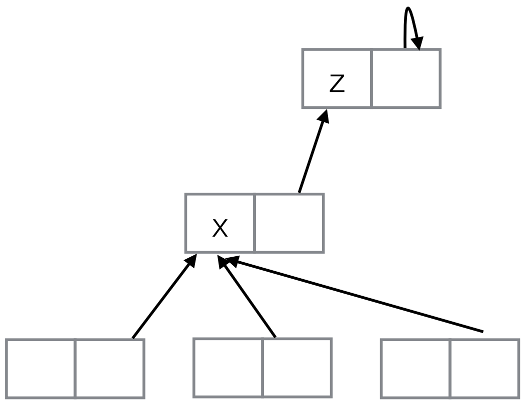
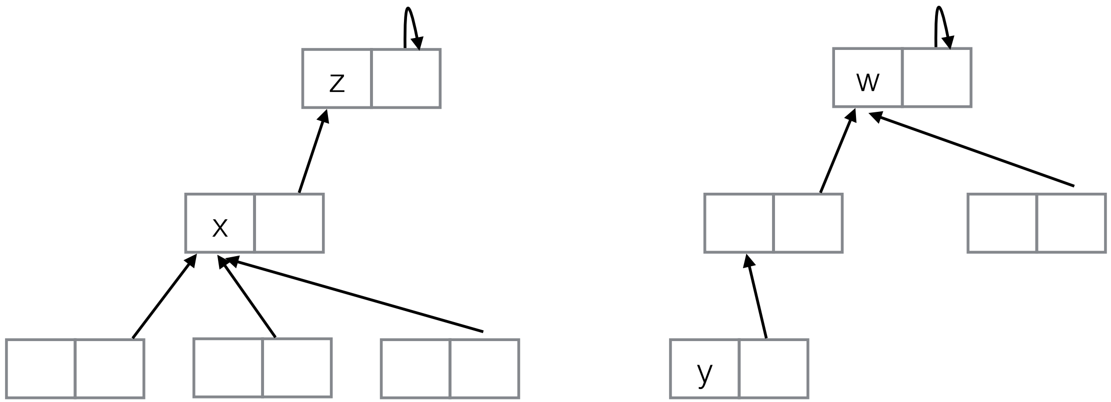



If $z \rightarrow \textbf{\textit{rank}} \geq w \rightarrow \textbf{\textit{rank}}$

# Union example

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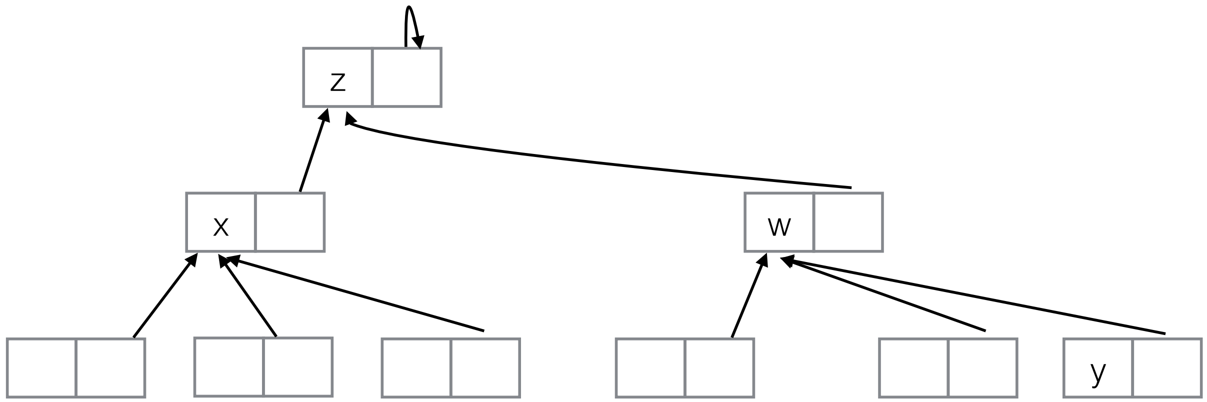

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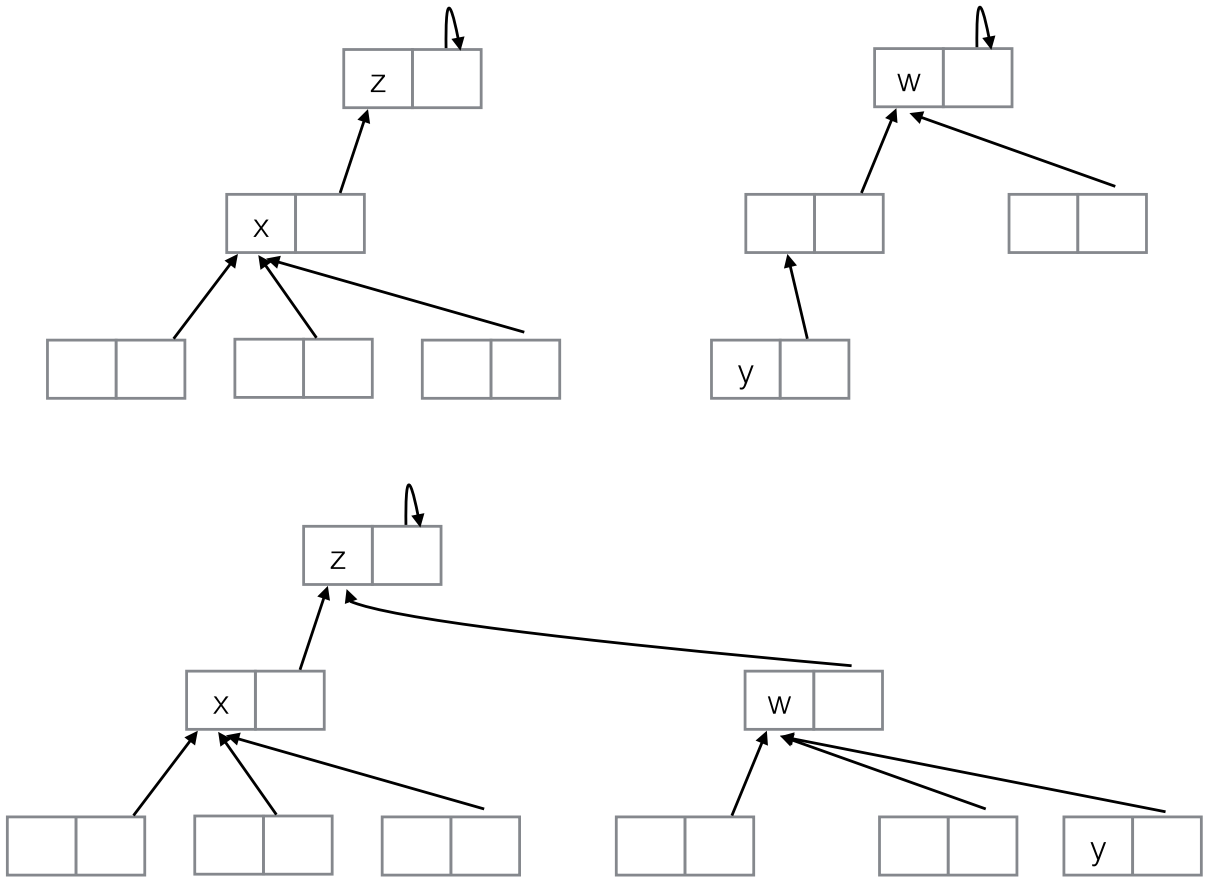


If $z \to \textit{rank} \geq w \to \textit{rank}$

If $z \to \textit{rank} = w \to \textit{rank}$, then $(z \to \textit{rank})++$

# Properties of Ranks

1. If $\boldsymbol{x}$ not a root, then $(\boldsymbol{x} \rightarrow \boldsymbol{rank}) < (\boldsymbol{x} \rightarrow \boldsymbol{parent} \rightarrow \boldsymbol{rank})$
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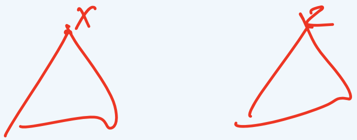

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$\Longrightarrow$ At least $2^{r-1} + 2^{r-1} = 2^r$ nodes in combined tree. □

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## Lemma

*There are at most $n/2^r$ nodes of rank at least $r$.*

## Proof.

Let $x$ node of rank at least $r$. Let $S_x$ be descendants of $x$ when it first got rank $r$.
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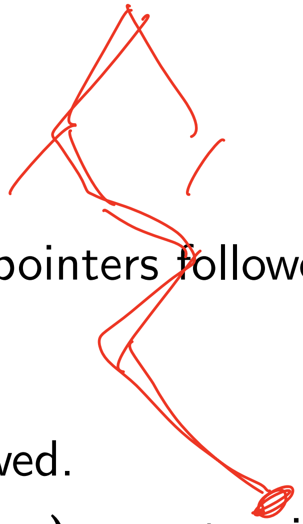

So at most $2m$ Finds, want to bound total # parent pointers followed.

- At most one parent pointer to root per Find $\implies$ at most $O(m)$ parent pointers to roots.
- So only need to worry about parent pointers to non-roots.

# Main Result II: Buckets

Put elements in buckets according to rank (only in analysis).

Notation: $\boldsymbol{2 \uparrow i}$ denote a tower of $\boldsymbol{i}$ **2**'s

- $\boldsymbol{2 \uparrow 1 = 2, \quad 2 \uparrow 2 = 2^2 = 4, \quad 2 \uparrow 3 = 2^{2^2} = 2^4 = 16, \quad 2 \uparrow 4 = 2^{2^{2^2}} = 2^{16} = 65536}$
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From Lemma: at most $n/(2^{2 \uparrow (i-1)}) = n/(2 \uparrow i)$ elements in bucket $i$.

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$$\sum_x \alpha(x) = \sum_{i=0}^{O(\log^* n)} \sum_{x \in B(i)} \alpha(x) \leq \sum_{i=0}^{O(\log^* n)} \sum_{x \in B(i)} (2 \uparrow i) \leq \sum_{i=0}^{O(\log^* n)} \frac{n}{2 \uparrow i}(2 \uparrow i) = O(n\log^* n)$$

$$\leq O(m\log^* n)$$