Lecture 10: Disjoint Sets / Union-Find

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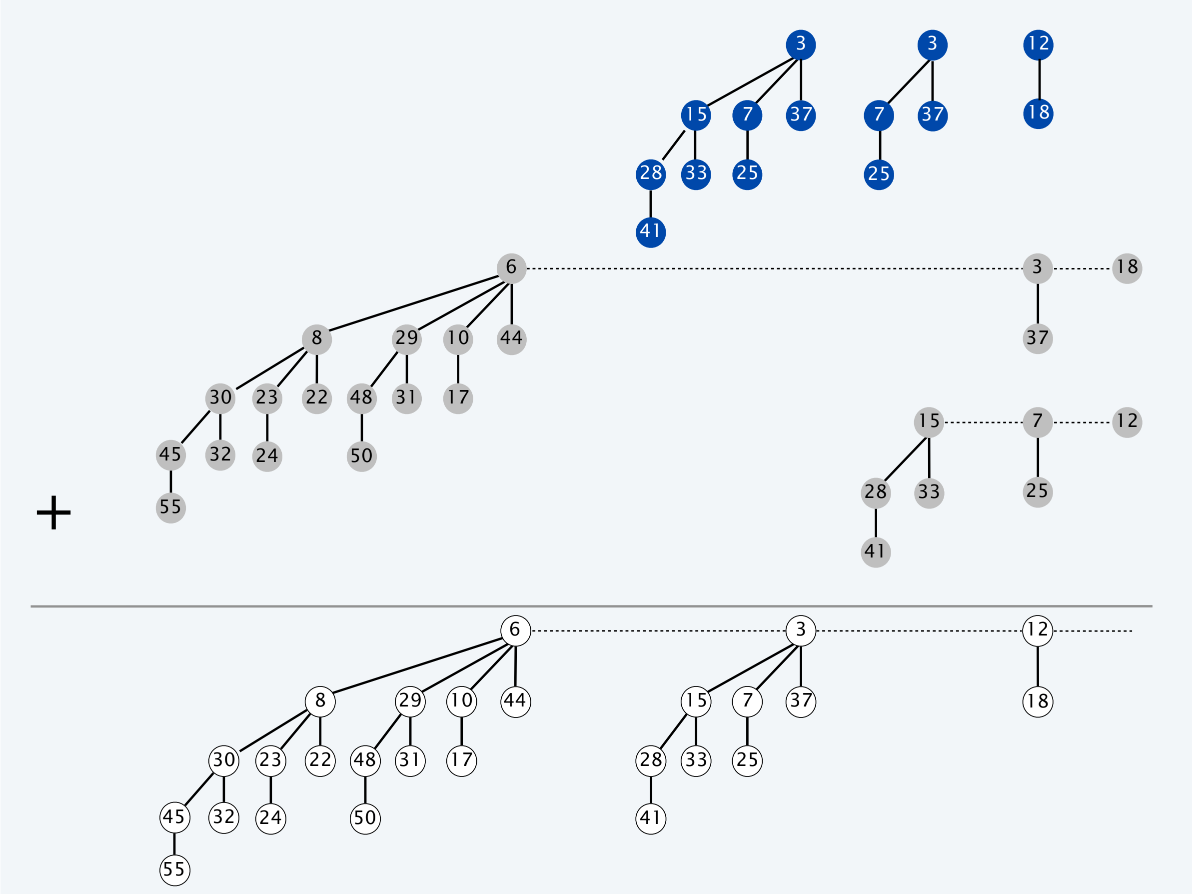
September 25, 2025
601.433/633 Introduction to Algorithms
Slides by Mike Dinitz

Meld(H_1, H_2): General Case

(Almost) just like binary addition!



Insert(H, x)

Use Meld:

- Create new heap H' with one B_0 consisting of just x
- Meld(H, H')

Correctness: Obvious

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Running Time:

- Worst case: $O(\log n)$ (via Meld)
- Amortized:
  - Like incrementing a binary counter!
  - If we link k trees, potential goes down by $k - 1$
  - Cost = # links plus 1 (for making new heap)
  - Amortized cost = $k + 1 + \Delta\Phi = k + 1 - (k - 1) = 2 = O(1)$

Extract-Min($\boldsymbol{H}$)

Use Meld again!

- $\boldsymbol{O(\log n)}$ to Find-Min: one of the roots.
- Delete and return this root: tree turns into a new heap!
- Meld with original heap (minus the tree)

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Running Time:

- Worst-Case: $O(\log n)$ from creating new heap, Meld
- Amortized:
  - Potential can go up! But by at most $\log n$
  - Amortized time at most $O(\log n) + \log n = O(\log n)$

Introduction to Union-Find

Informal: Universe of elements, want to maintain *disjoint sets*.

Slightly more formally:

- Make-Set(x): create a new set containing just x (i.e., $\{x\}$)
- Union(x, y): Replace set containing x (S) and set containing y (T) with single set $S \cup T$
- Find(x): Return *representative* of set containing x

For $x, y \in S$

Find(x) $\stackrel{?}{=}$ Find(y)

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Rules: every set has a *unique* representative.

- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in same set, Find($\boldsymbol{x}$) = Find($\boldsymbol{y}$)
- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in different sets, then Find($\boldsymbol{x}$) $\neq$ Find($\boldsymbol{y}$)
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Note: disjoint (and partition) by construction!

Introduction (II)

We'll see a few ways of doing this, from efficient to very efficient.
CLRS: extremely efficient

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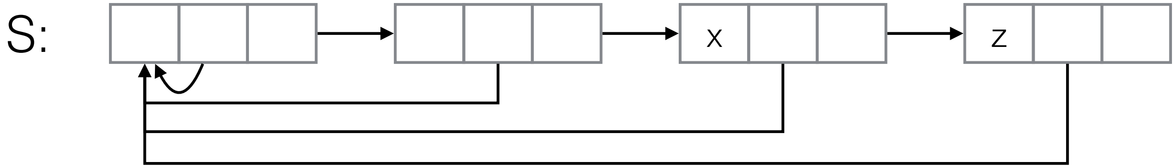
Notation and Notes:

- $\boldsymbol{m}$ operations total
- $\boldsymbol{n}$ of which are Make-Sets (so $\boldsymbol{n}$ elements)
- Assume have pointer/access to elements we care about (like last class)

First Approach: Lists

Linked list for each set.

- Representative of set is head (first element on list)
- Each element has pointer to head and to next element, so stored as triple: (element, head, next)



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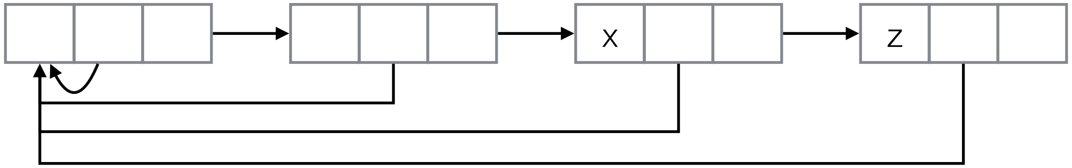
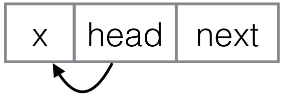


Make-Set(x):



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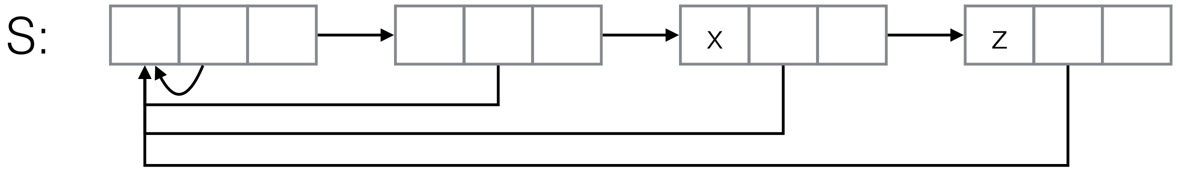

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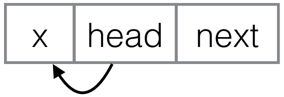
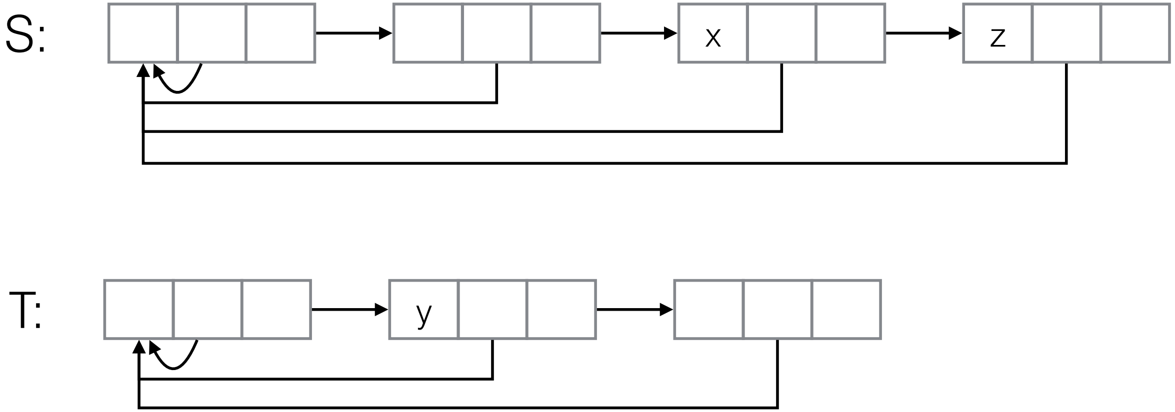


Find(***x***): return ***x*** → head

Union(x, y)



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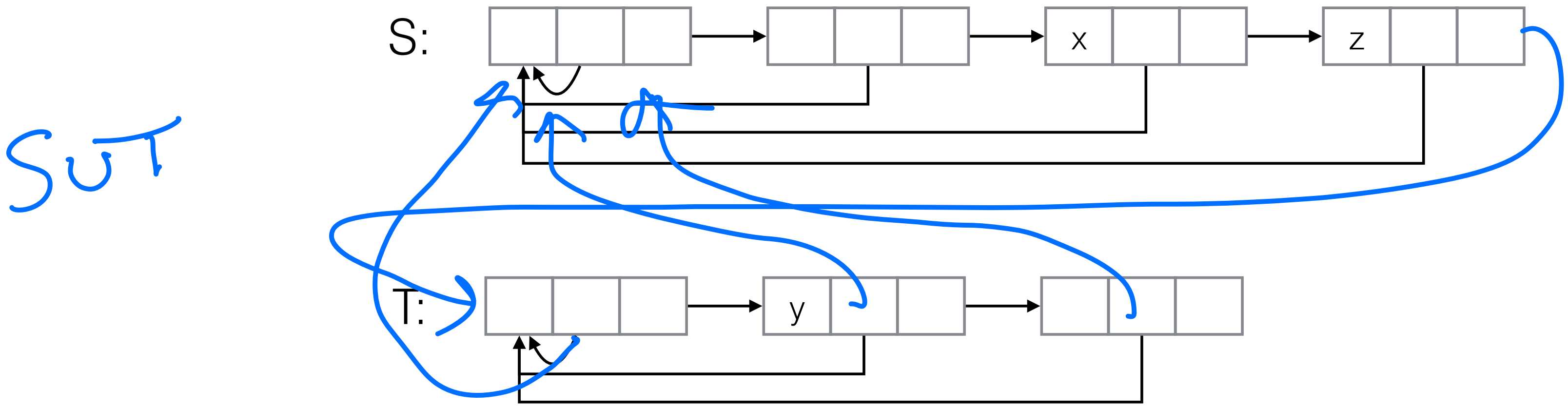
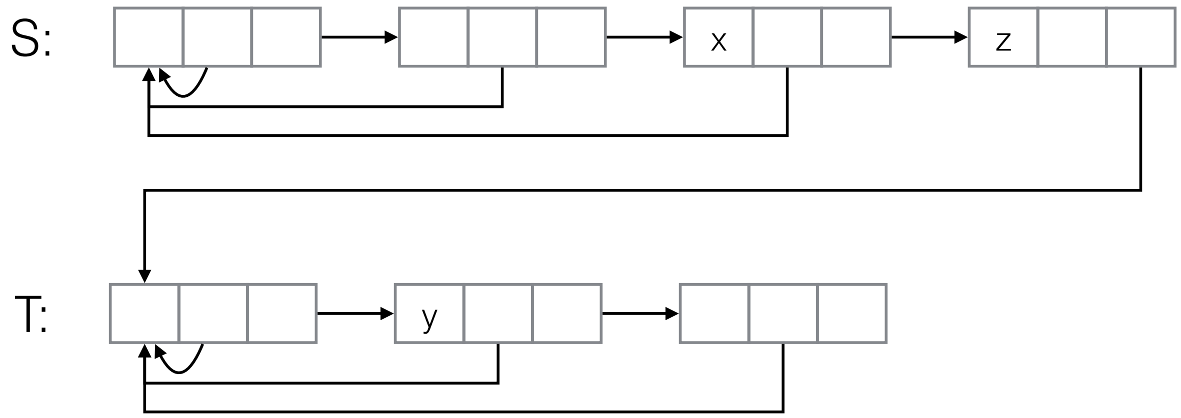


Obvious approach:

- Walk down S to final element z (starting from x)
- Set $z \rightarrow$ next = $y \rightarrow$ head
- Walk down T, set every elements head pointer to $x \rightarrow$ head

Union($\boldsymbol{x}, \boldsymbol{y}$)



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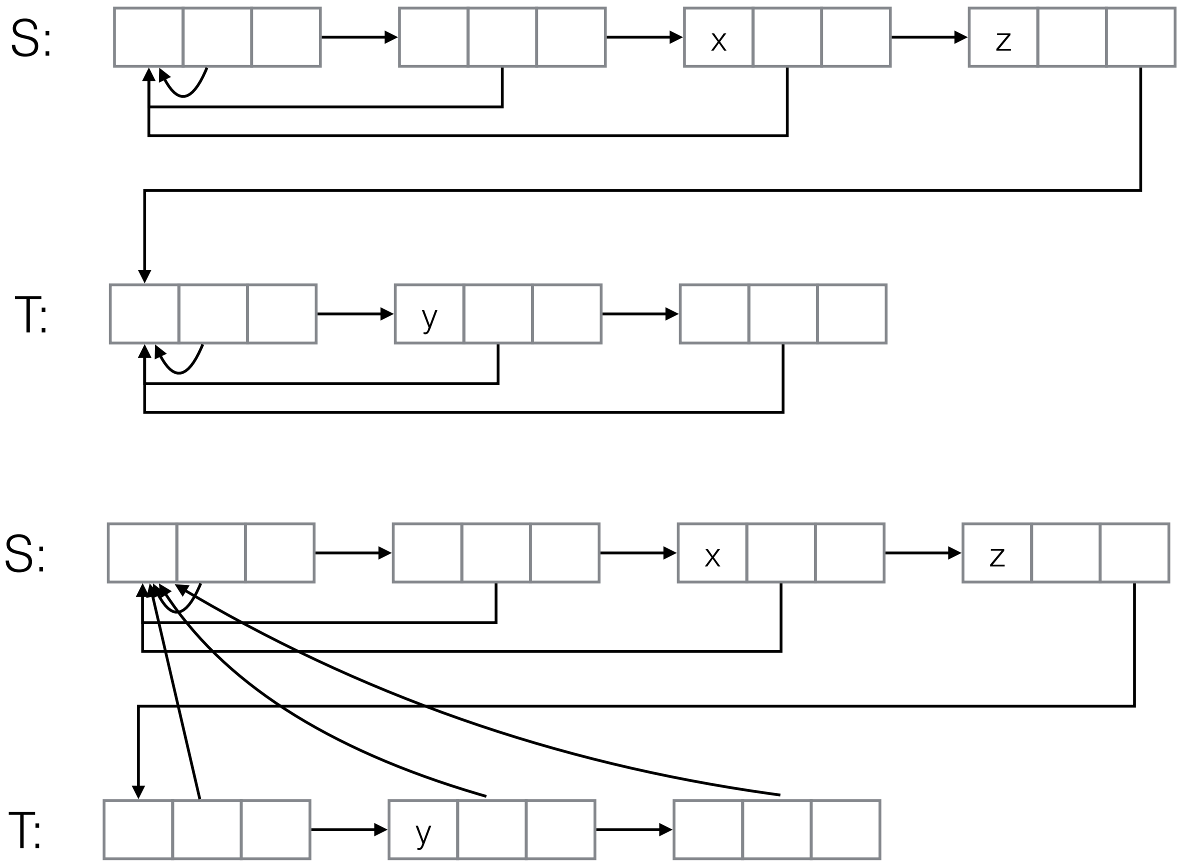

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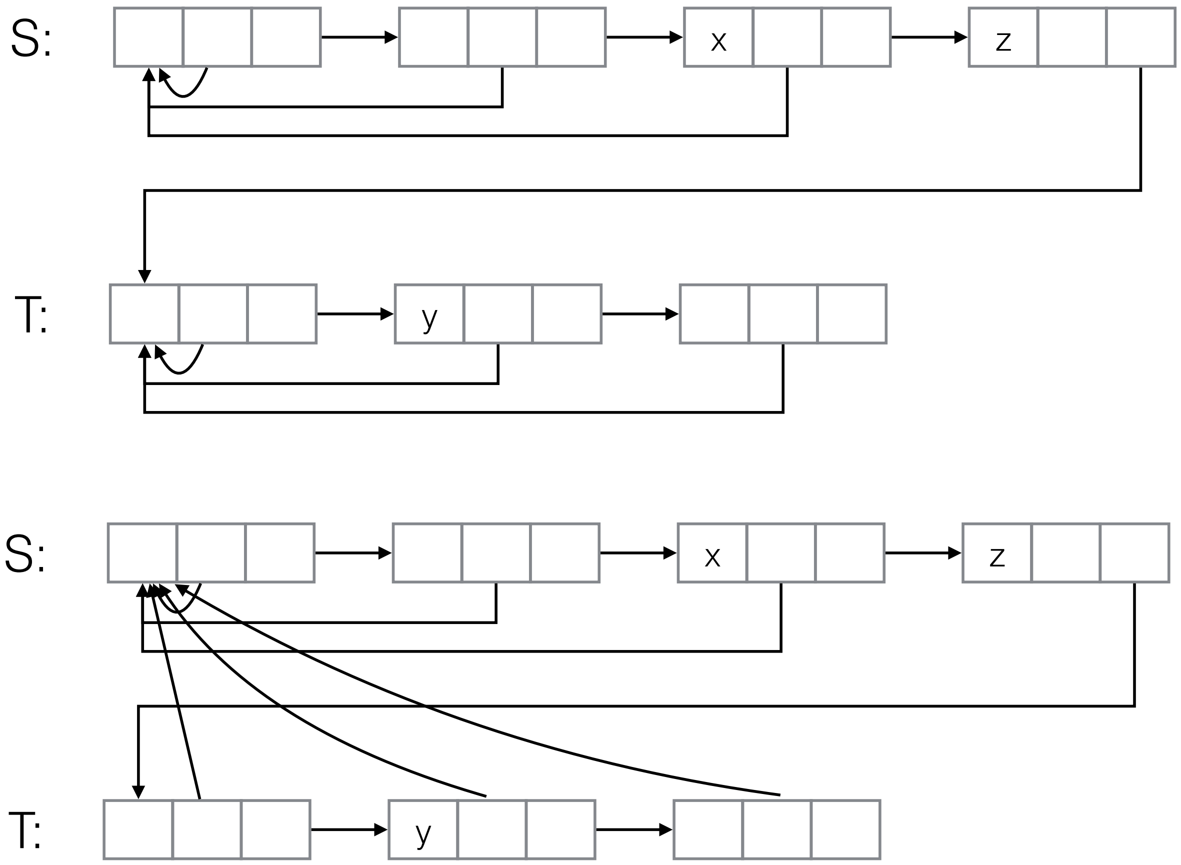
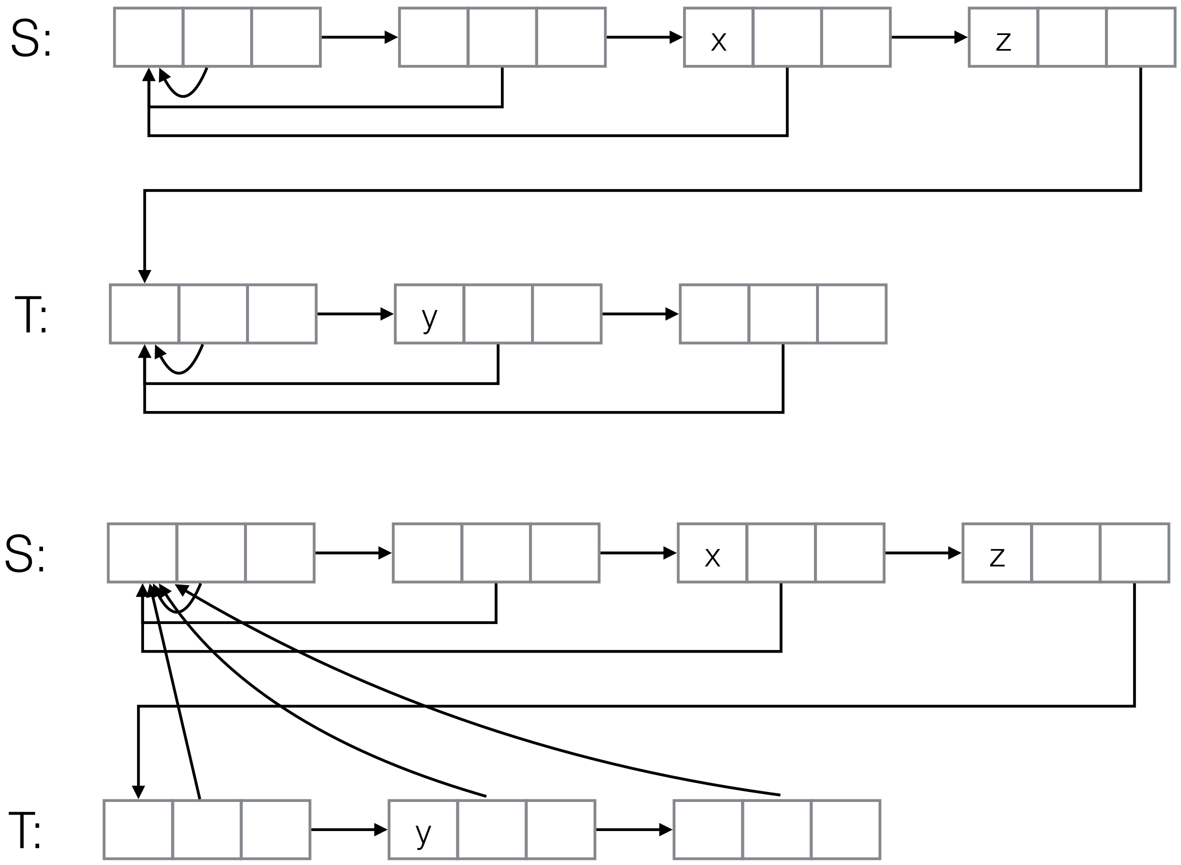


Running time:

Union(x, y)



Running time: $O(|S| + |T|)$

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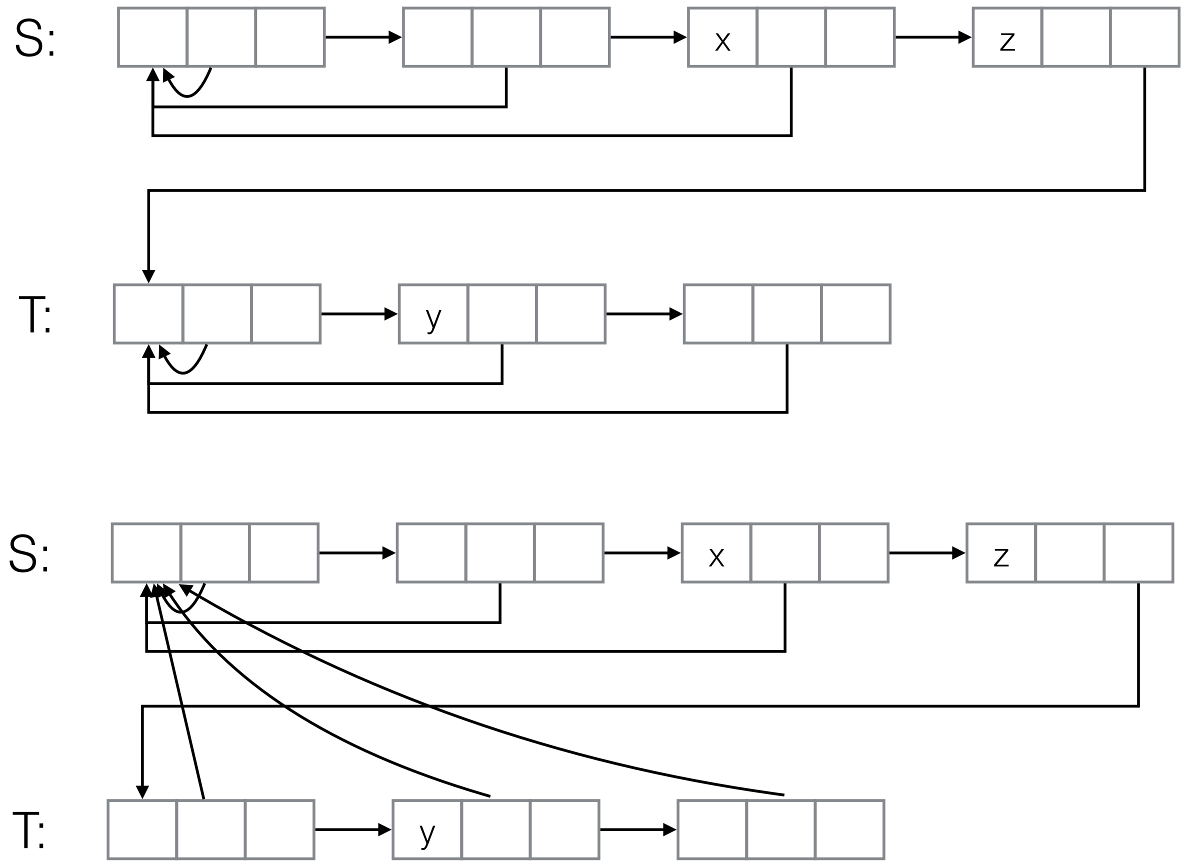

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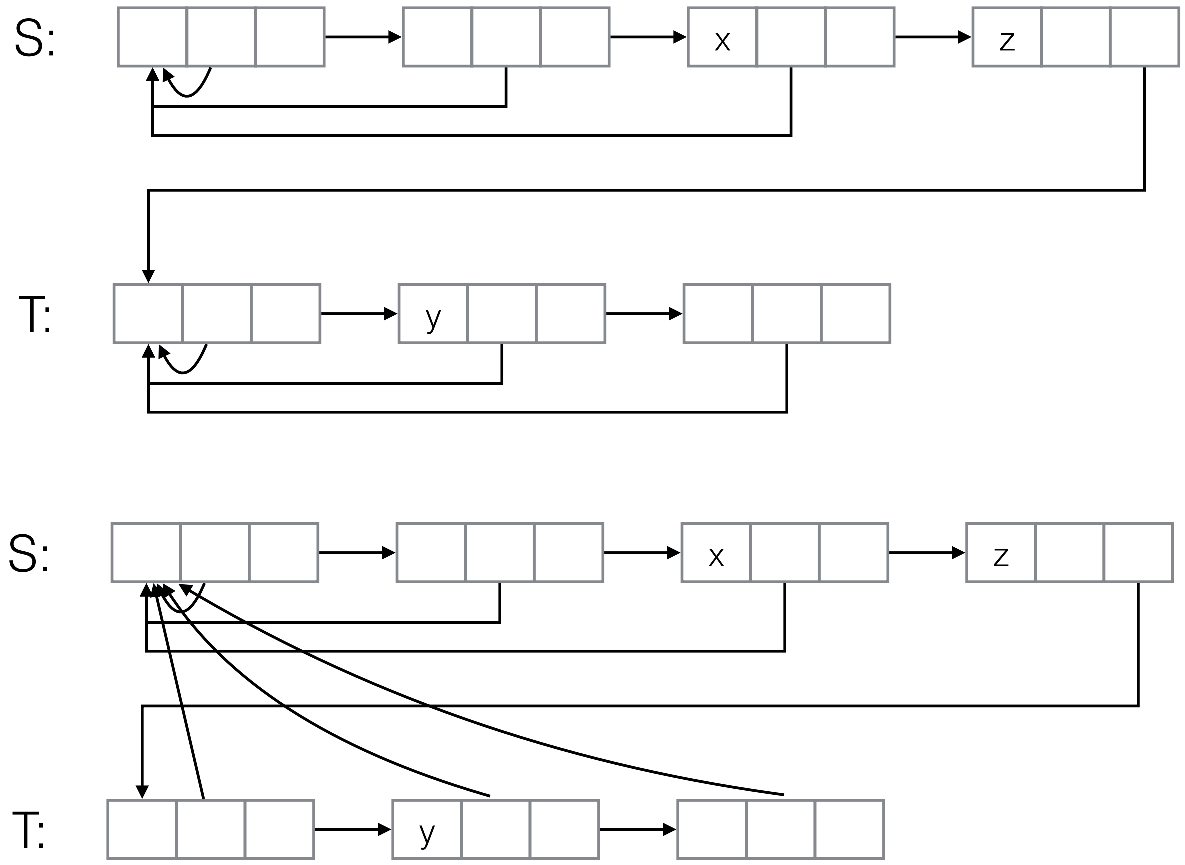

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Since $|S|, |T|$ could be $\Theta(n)$, can only say $O(n)$ for Unions

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Observation: don't need to preserve ordering inside the Union!

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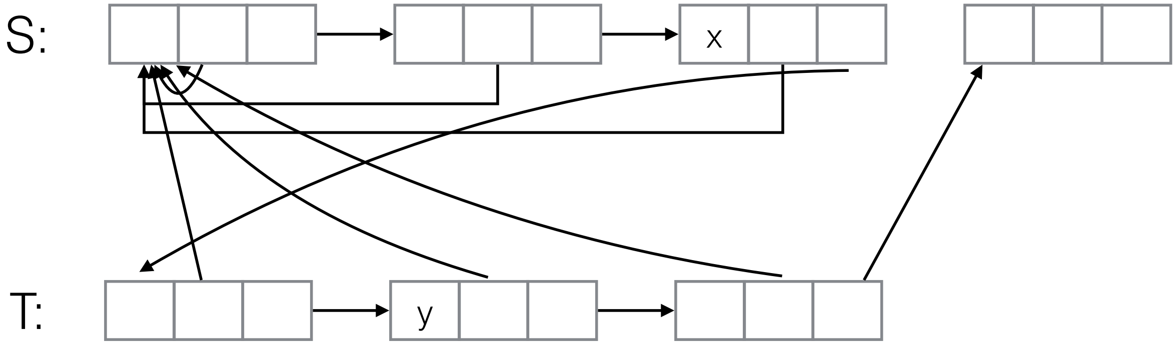

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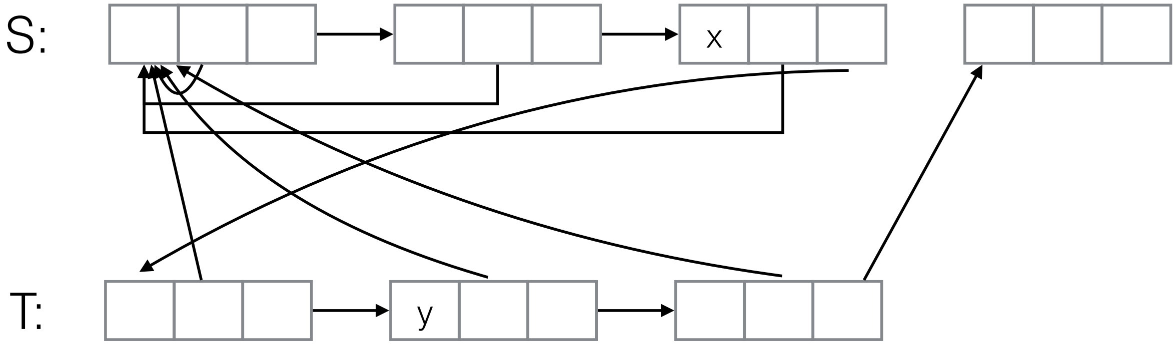

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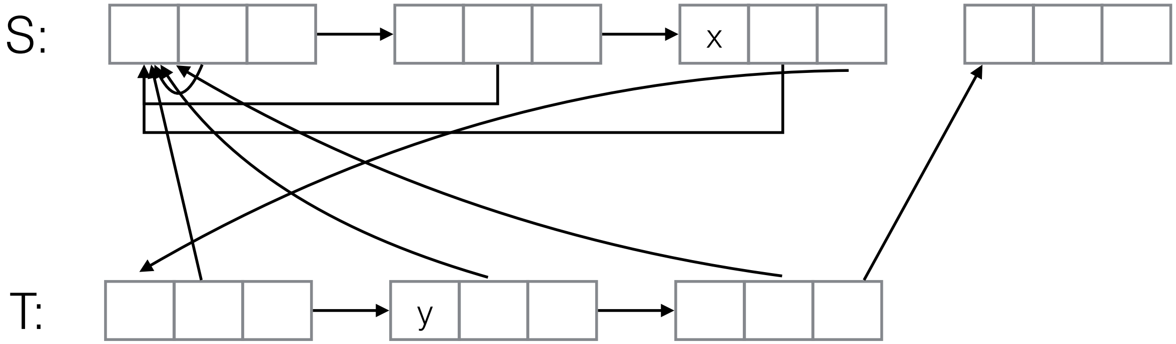

Running time: $O(|T|)$

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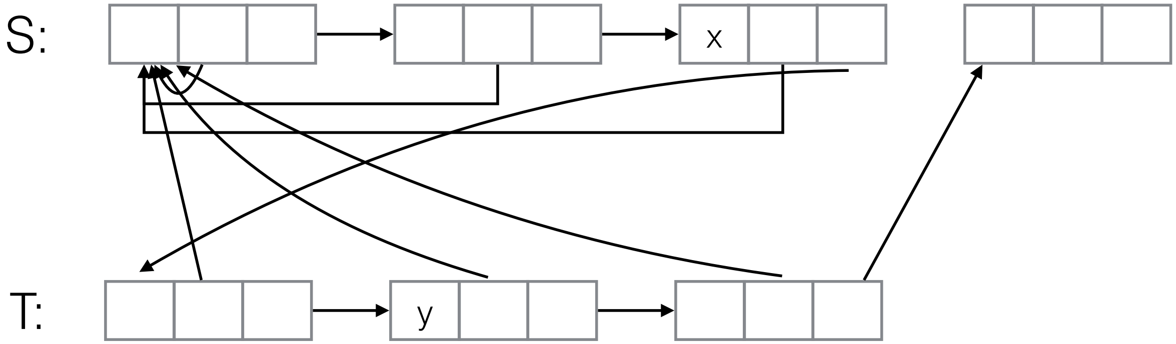

Running time: $O(|T|)$

- Still can't say anything better than $O(n)$

Even more improved Union(x, y)

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Theorem

The amortized cost of Find and Union is $O(1)$, and the amortized cost of Make-Set is $O(\log n)$.

Corollary

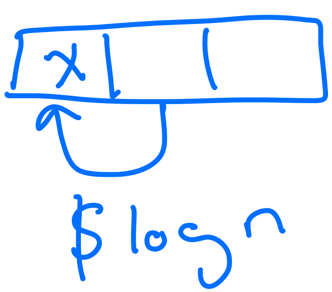
The total running time is $O(m + n \log n)$.

Amortized Analysis of List Algorithm

Banking/accounting argument: bank for every element

- When an element is created (via Make-Set), add $\log n$ tokens to its bank
- Find does not affect banks
- When doing Union, take token from bank of each element in smaller set.

MakeSet(x) :



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- Can only happen at most $\log n$ times.

□

Amortized Analysis of List Algorithm (cont'd)

Make-Set:

- True cost: $O(1)$
- Change in banks: $\log n$

$\implies$ Amortized cost: $O(1) + O(\log n) = O(\log n)$

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Find:

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- Change in banks: 0

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$O(n \log n + m)$

Find:

- True cost: $O(1)$
- Change in banks: 0

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$O(m \log^* n)$

Union:

- True cost: $\min(|S|, |T|)$
- Change in banks: $-\min(|S|, |T|)$

$\implies$ Amortized cost: $\min(|S|, |T|) - \min(|S|, |T|) = 0 = O(1)$.

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Starting idea: want to make Unions faster, willing to make Finds a little slower.

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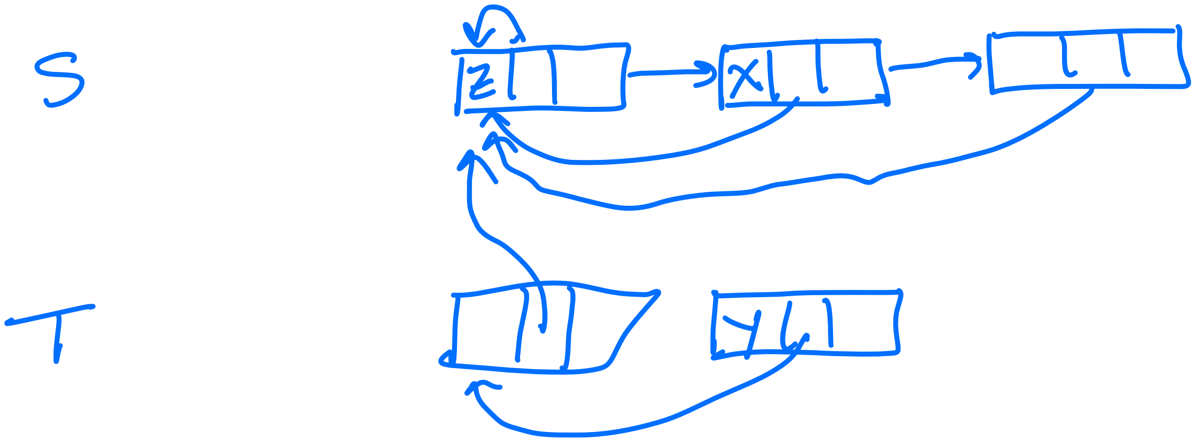

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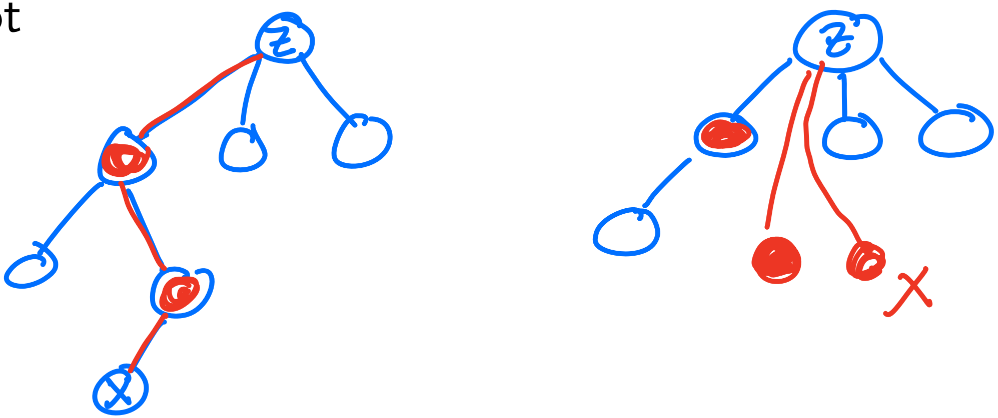

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Idea 2: *Union By Rank*

- Size of set was important for lists, less important for trees.
- Choose which set to splice into which by *rank* of trees (related to height)

Main Result

Theorem

When using Path Compression and Union By Rank, total time at most $O(m \log^* n)$.

$\log^*$: iterated $\log_2$.

- $\log^* n$ = # times apply $\log_2$ until get to ≤ 1

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Stronger theorem: total time at most $O(m \cdot \alpha(m, n))$.

- $\alpha(m, n)$: inverse Ackermann function. Grows even slower than $\log^*$.
- See CLRS for details

Formal Procedures: Make-Set and Find

Make-Set(x): Set $x \rightarrow rank = 0$ and $x \rightarrow parent = x$

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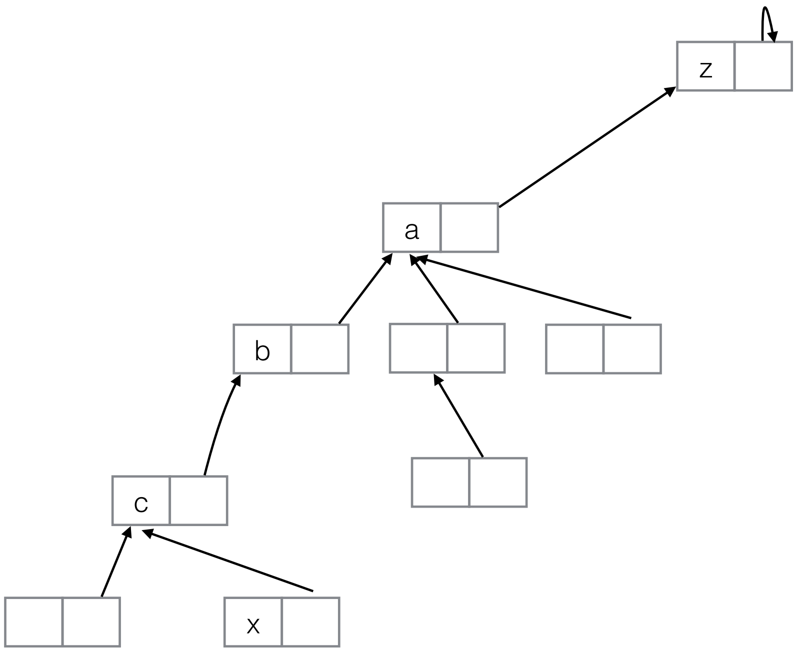
Running time of Find: depth of x (distance to root)

Find example



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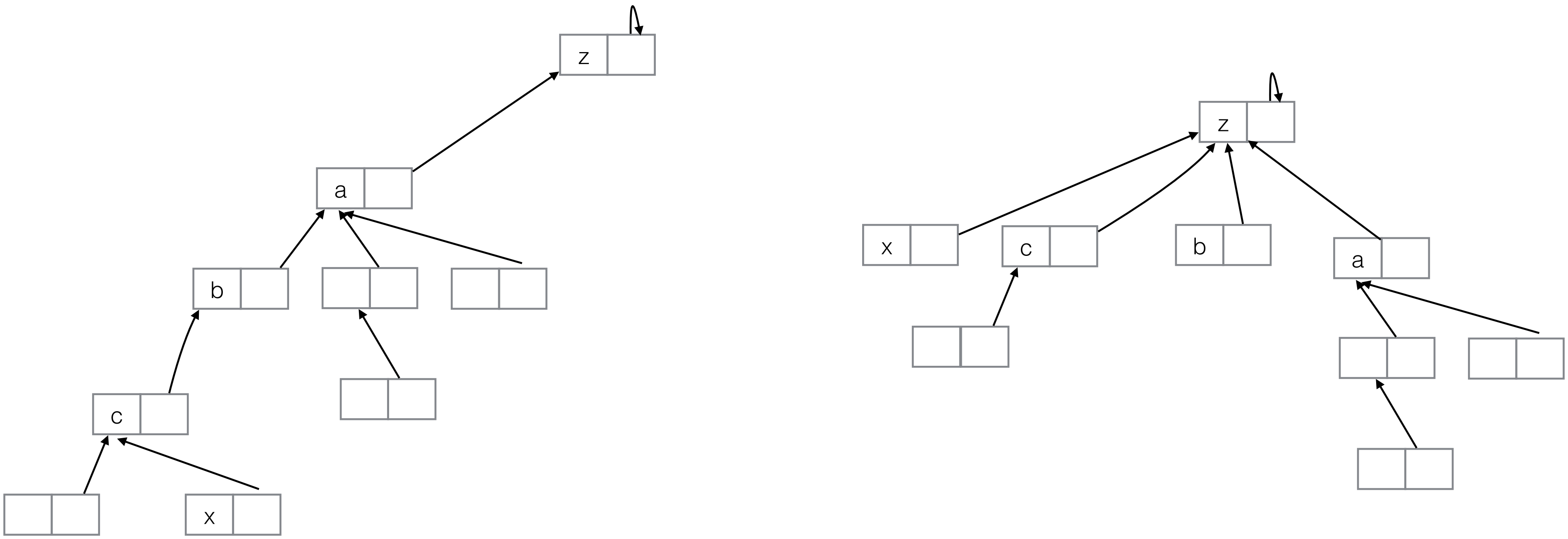

Formal Procedure: Union

Link(r_1, r_2): Only applied to root nodes

- If $r_1 \to \textit{\textbf{rank}} > r_2 \to \textit{\textbf{rank}}$, set $r_2 \to \textit{\textbf{parent}} = r_1$
- If $r_2 \to \textit{\textbf{rank}} > r_1 \to \textit{\textbf{rank}}$, set $r_1 \to \textit{\textbf{parent}} = r_2$
- If $r_1 \to \textit{\textbf{rank}} = r_2 \to \textit{\textbf{rank}}$, set $r_2 \to \textit{\textbf{parent}} = r_1$ and increment $r_1 \to \textit{\textbf{rank}}$.

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Union(x, y): Link(Find(x), Find(y))

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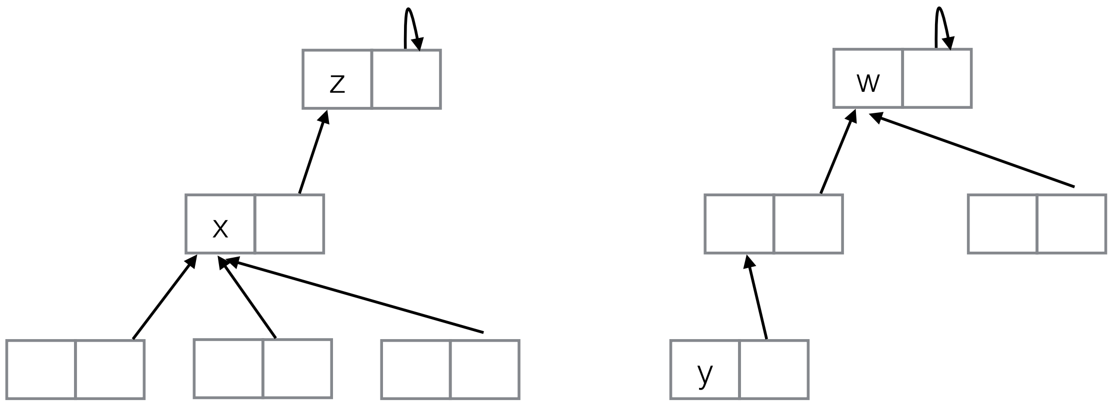
Union(x, y): Link(Find(x), Find(y))

- Running time: depth(x) + depth(y)

Union example



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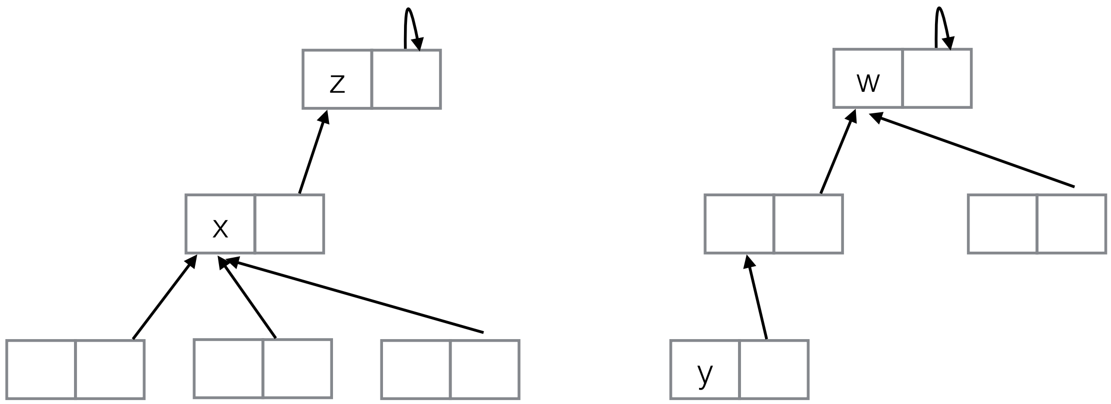

If $\boldsymbol{z \rightarrow rank \geq w \rightarrow rank}$

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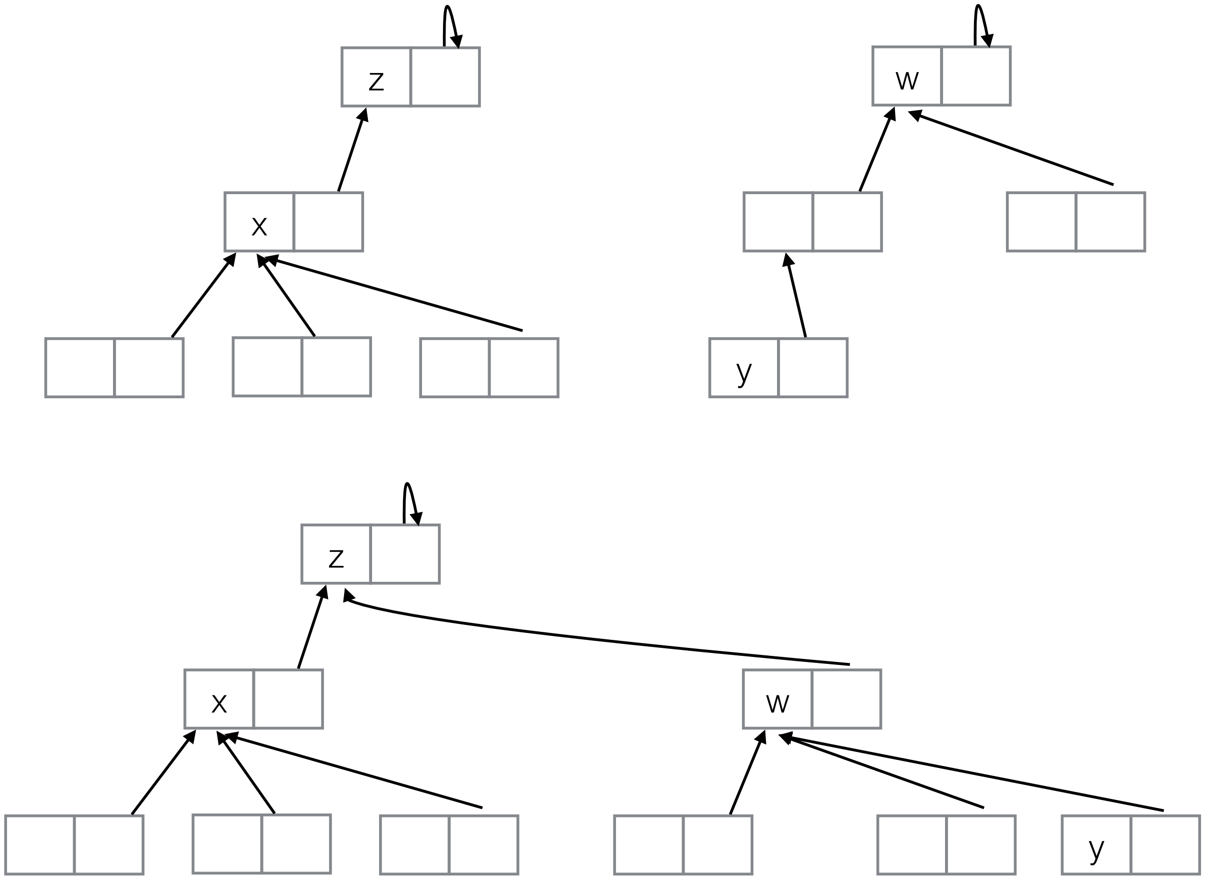

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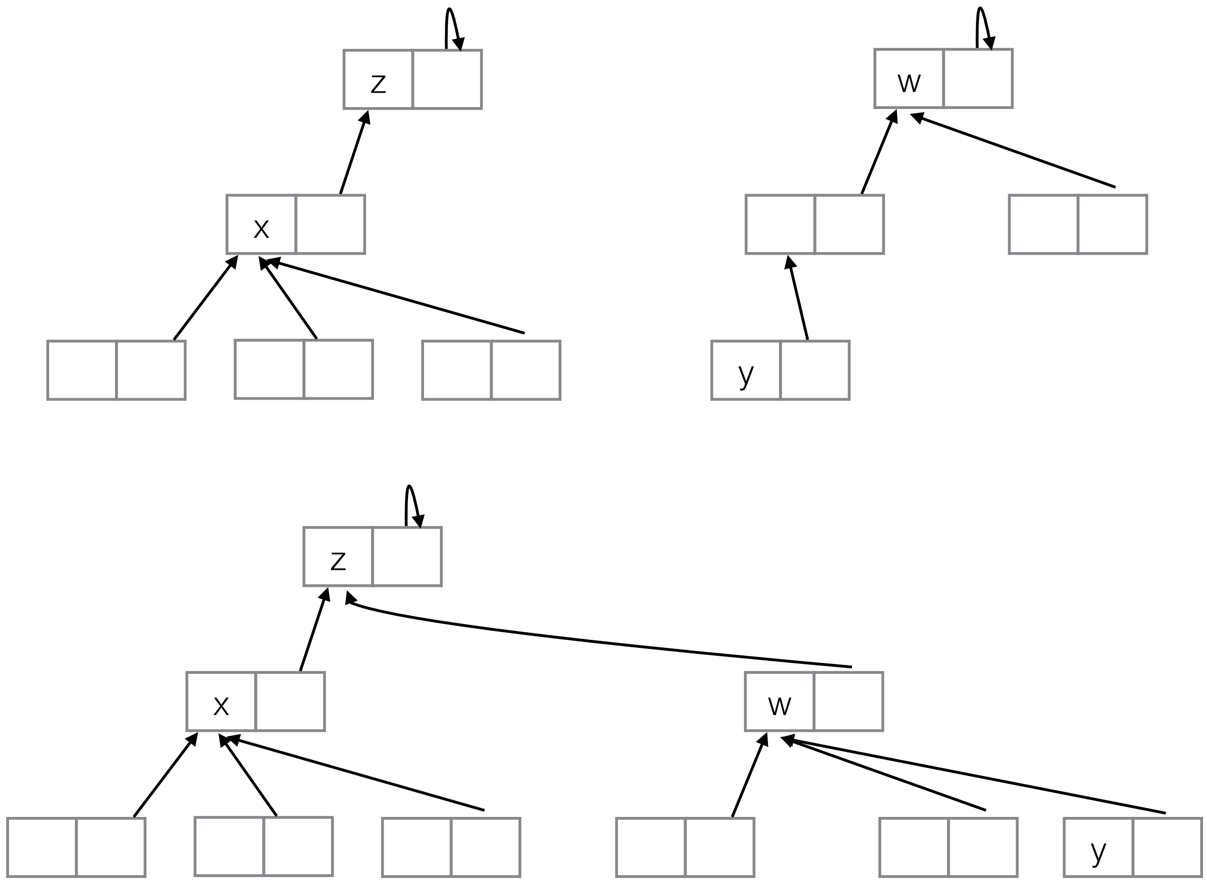

If $z \rightarrow \textbf{\textit{rank}} \geq w \rightarrow \textbf{\textit{rank}}$

If $z \rightarrow \textbf{\textit{rank}} = w \rightarrow \textbf{\textit{rank}}$,
then $(z \rightarrow \textbf{\textit{rank}})++$

Properties of Ranks

1. If $\boldsymbol{x}$ not a root, then $(\boldsymbol{x} \rightarrow \boldsymbol{rank}) < (\boldsymbol{x} \rightarrow \boldsymbol{parent} \rightarrow \boldsymbol{rank})$
2. When doing path compression, if parent of $\boldsymbol{x}$ changes, new parent has rank strictly larger than old parent
3. $\boldsymbol{x} \rightarrow \boldsymbol{rank}$ can change only if $\boldsymbol{x}$ a root, and once $\boldsymbol{x}$ is a non-root it never becomes a root again.

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$\implies$ By induction, at least 2^{r-1} nodes in each tree

$\implies$ At least $2^{r-1} + 2^{r-1} = 2^r$ nodes in combined tree. □

Nodes of rank r

Lemma

There are at most $n/2^r$ nodes of rank at least r.

Proof.

Let x node of rank at least r. Let S_x be descendants of x when it first got rank r.
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So at most $2m$ Finds, want to bound total # parent pointers followed.

- At most one parent pointer to root per Find $\implies$ at most $O(m)$ parent pointers to roots.
- So only need to worry about parent pointers to non-roots.

Main Result II: Buckets

Put elements in buckets according to rank (only in analysis).

Notation: $\mathbf{2} \uparrow \boldsymbol{i}$ denote a tower of $\boldsymbol{i}$ $\mathbf{2}$'s

- $\mathbf{2 \uparrow 1 = 2}, \quad \mathbf{2 \uparrow 2 = 2^2 = 4}, \quad \mathbf{2 \uparrow 3 = 2^{2^2} = 2^4 = 16}, \quad \mathbf{2 \uparrow 4 = 2^{2^{2^2}} = 2^{16} = 65536}$
- $\mathbf{\log^*(2 \uparrow \boldsymbol{i}) = \boldsymbol{i}}$

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$\boldsymbol{B(i)}$ (Bucket $\boldsymbol{i}$): All elements of rank at least $\mathbf{2 \uparrow (\boldsymbol{i} - 1)}$, at most $\mathbf{(2 \uparrow \boldsymbol{i}) - 1}$

- Bucket $\mathbf{0}$: nodes with rank $\mathbf{0}$
- Bucket $\mathbf{1}$: rank at least $\mathbf{1}$, at most $\mathbf{1}$
- Bucket $\mathbf{2}$: rank at least $\mathbf{2}$, at most $\mathbf{3}$
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- $\log^*(2 \uparrow i) = i$

$B(i)$ (Bucket i): All elements of rank at least $2 \uparrow (i-1)$, at most $(2 \uparrow i) - 1$

- Bucket 0: nodes with rank 0
- Bucket 1: rank at least 1, at most 1
- Bucket 2: rank at least 2, at most 3
- Bucket 3: rank at least 4, at most 15
- Bucket 4: rank at least 16, at most 65535
- At most $\log^* n$ buckets.

From Lemma: at most $n/(2^{2\uparrow(i-1)}) = n/(2 \uparrow i)$ elements in bucket i.

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- For each x, let $\alpha(x) = \#$ times follow parent point from x to parent in same bucket, not root. Want to show $\sum_x \alpha(x) \leq O(m \log^* n)$.
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$$\sum_x \alpha(x) = \sum_{i=0}^{O(\log^* n)} \sum_{x \in B(i)} \alpha(x) \leq \sum_{i=0}^{O(\log^* n)} \sum_{x \in B(i)} (2 \uparrow i) \leq \sum_{i=0}^{O(\log^* n)} \frac{n}{2 \uparrow i}(2 \uparrow i) = O(n \log^* n)$$

$$\leq O(m \log^* n)$$