Lecture 9: Disjoint Sets / Union-Find

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September 25, 2025
601.433/633 Introduction to Algorithms

Introduction

Informal: Universe of elements, want to maintain *disjoint sets*.

Slightly more formally:

- Make-Set($\boldsymbol{x}$): create a new set containing just $\boldsymbol{x}$ (i.e., $\{\boldsymbol{x}\}$)
- Union($\boldsymbol{x}, \boldsymbol{y}$): Replace set containing $\boldsymbol{x}$ ($\boldsymbol{S}$) and set containing $\boldsymbol{y}$ ($\boldsymbol{T}$) with single set $\boldsymbol{S} \cup \boldsymbol{T}$
- Find($\boldsymbol{x}$): Return *representative* of set containing $\boldsymbol{x}$

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Rules: every set has a *unique* representative.

- If x and y are in same set, Find(x) = Find(y)
- If x and y are in different sets, then Find(x) $\neq$ Find(y)
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Note: disjoint (and partition) by construction!

Introduction (II)

We'll see a few ways of doing this, from efficient to very efficient.
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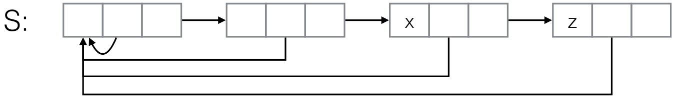
Notation and Notes:

- m operations total
- n of which are Make-Sets (so n elements)
- Assume have pointer/access to elements we care about (like last class)

First Approach: Lists

Linked list for each set.

- Representative of set is head (first element on list)
- Each element has pointer to head and to next element, so stored as triple: (element, head, next)



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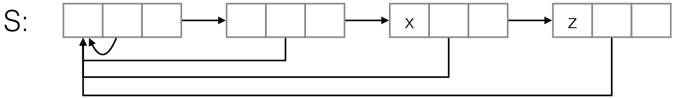

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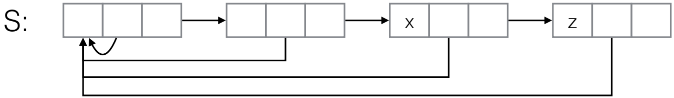
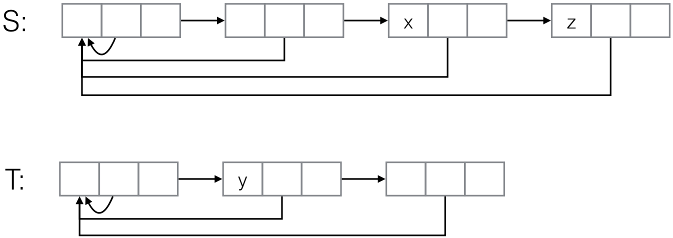


Make-Set(x):

Find(x): return $x \rightarrow$ head

Union(x, y)



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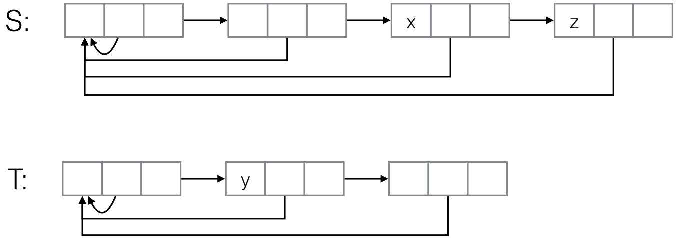

Obvious approach:

- Walk down S to final element z (starting from x)
- Set $z \to \text{next} = y \to \text{head}$
- Walk down T, set every elements head pointer to $x \to \text{head}$

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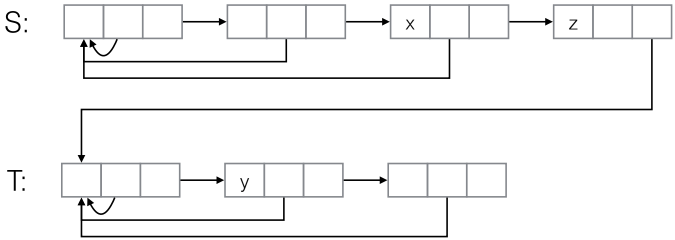

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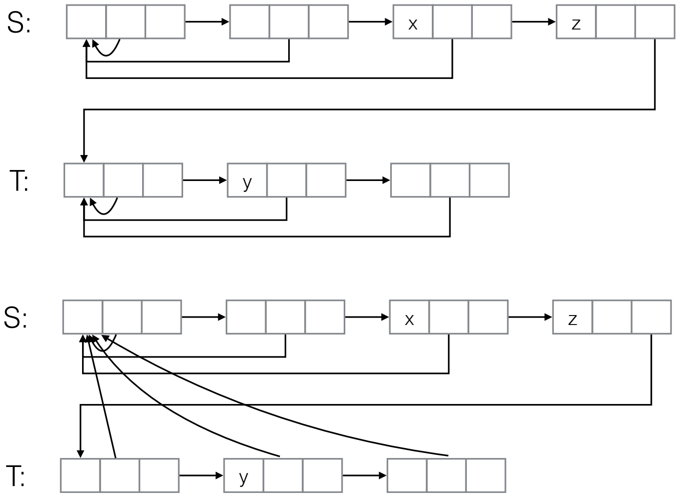

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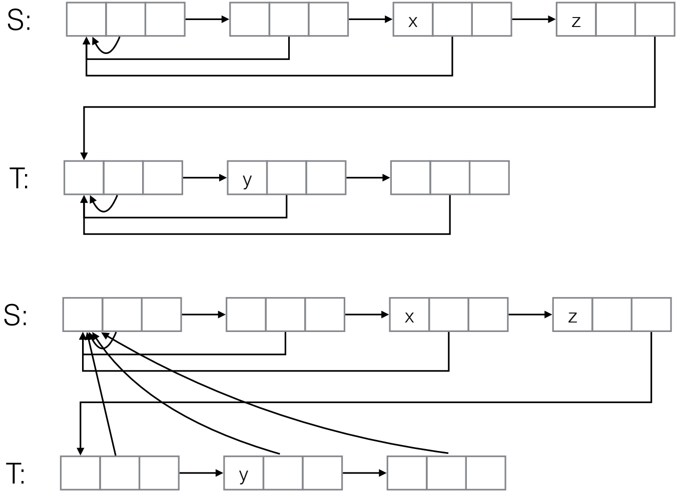
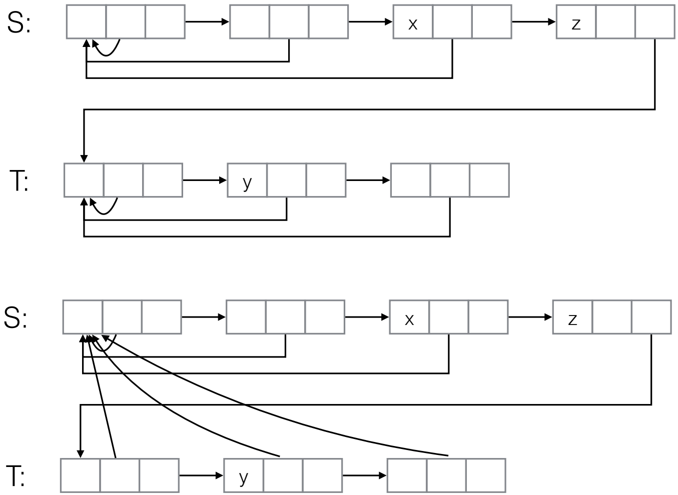


Running time:

Union(x, y)



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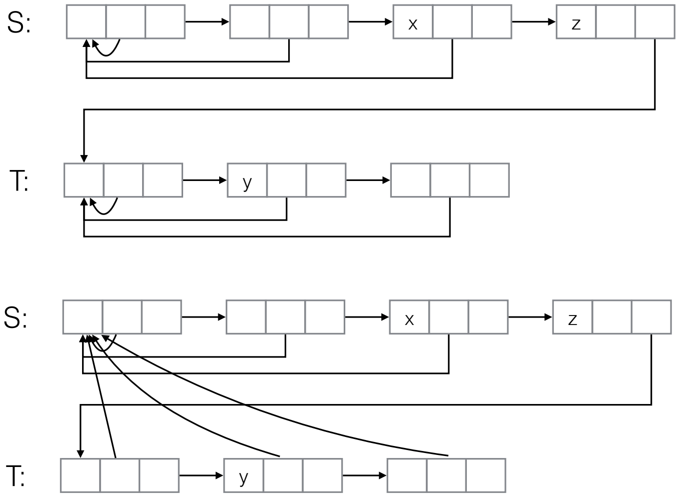

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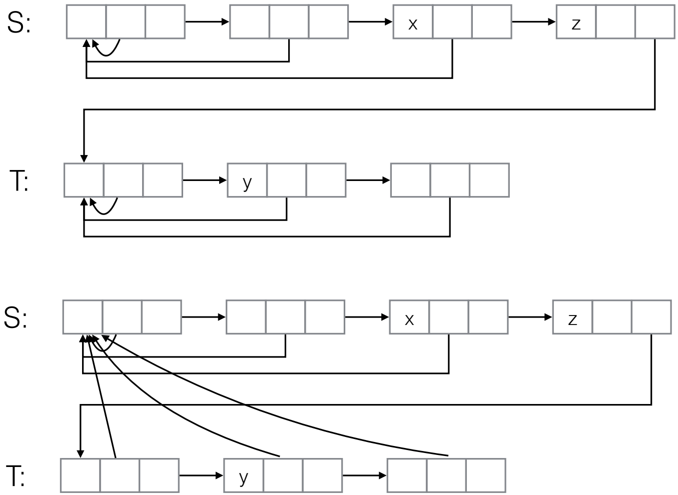

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Since $|S|, |T|$ could be $\Theta(n)$, can only say $O(n)$ for Unions

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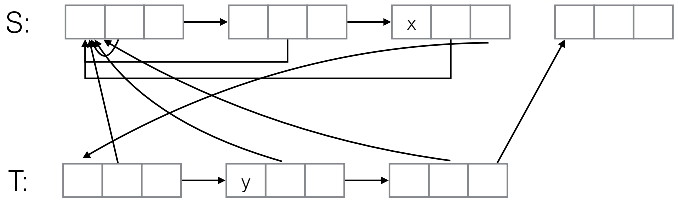

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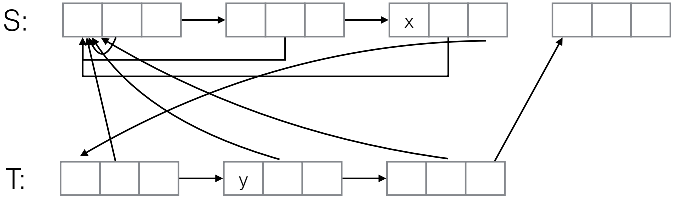

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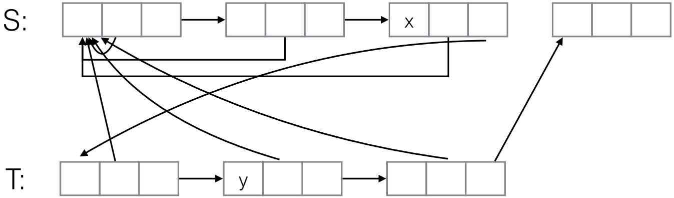

Running time: $O(|T|)$

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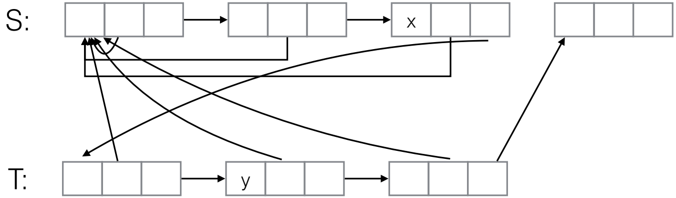

Running time: $O(|T|)$

- Still can't say anything better than $O(n)$

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Theorem

The amortized cost of Find and Union is $O(1)$, and the amortized cost of Make-Set is $O(\log n)$.

Corollary

The total running time is $O(m + n \log n)$.

Amortized Analysis of List Algorithm

Banking/accounting argument: bank for every element

- When an element is created (via Make-Set), add $\log n$ tokens to its bank
- Find does not affect banks
- When doing Union, take token from bank of each element in smaller set.

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- Size of set containing e at least doubles!
- Can only happen at most $\log n$ times. $\square$

Amortized Analysis of List Algorithm (cont'd)

Make-Set:

- True cost: $O(1)$
- Change in banks: $\log n$

$\Longrightarrow$ Amortized cost: $O(1) + O(\log n) = O(\log n)$

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Union:

- True cost: $\min(|S|, |T|)$
- Change in banks: $-\min(|S|, |T|)$

$\Longrightarrow$ Amortized cost: $\min(|S|, |T|) - \min(|S|, |T|) = 0 = O(1)$.

Even Better

Starting idea: want to make Unions faster, willing to make Finds a little slower.

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- *Path Compression*

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Idea 2: *Union By Rank*

- Size of set was important for lists, less important for trees.
- Choose which set to splice into which by *rank* of trees (related to height)

Main Result

Theorem

When using Path Compression and Union By Rank, total time at most $O(m \log^* n)$.

$\log^*$: iterated $\log_2$.

- $\log^* n = \#$ times apply $\log_2$ until get to 1

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Stronger theorem: total time at most $O(m \cdot \alpha(m,n))$.

- $\alpha(m,n)$: inverse Ackermann function. Grows even slower than $\log^*$.
- See CLRS for details

Formal Procedures: Make-Set and Find

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Find(x): Walk from x to root, and return root. Set parent pointers of x and all ancestors to root.

- If $x \rightarrow \textit{parent} = x$ then return x
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- Return $x \rightarrow \textit{parent}$

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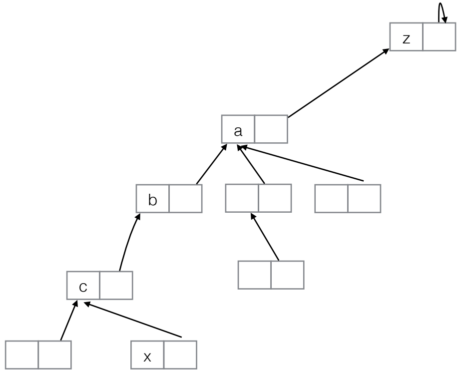
Running time of Find: depth of x (distance to root)

Find example



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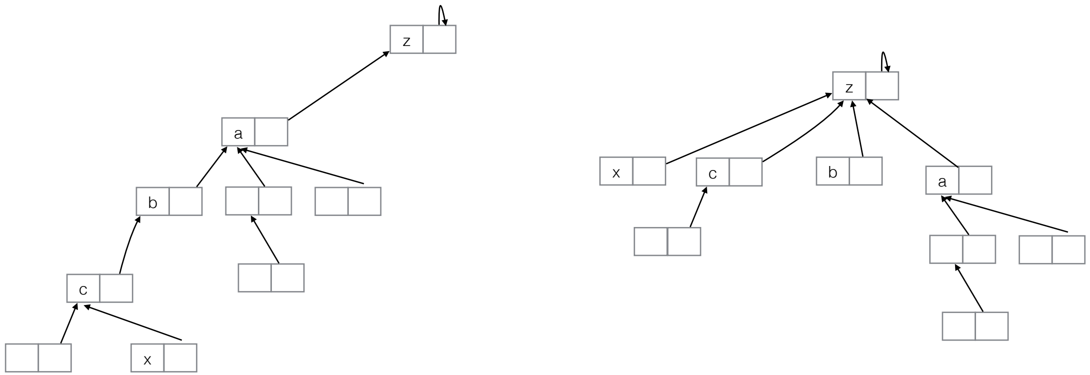

Formal Procedure: Union

Link($\boldsymbol{r_1}, \boldsymbol{r_2}$): Only applied to root nodes

- If $\boldsymbol{r_1 \rightarrow rank} > \boldsymbol{r_2 \rightarrow rank}$, set $\boldsymbol{r_2 \rightarrow parent} = \boldsymbol{r_1}$
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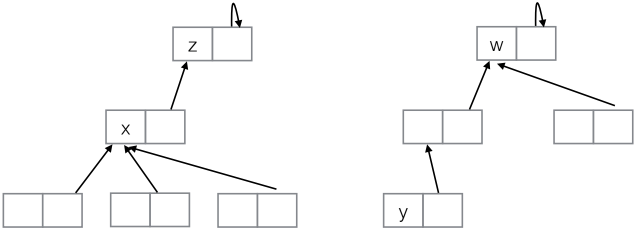
Union(x, y): Link(Find(x), Find(y))

- Running time: depth(x) + depth(y)

Union example



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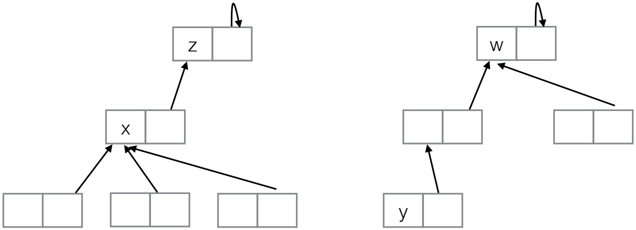

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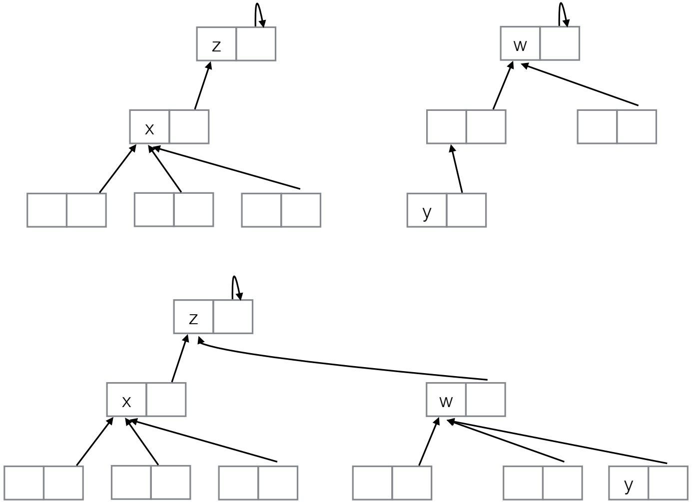

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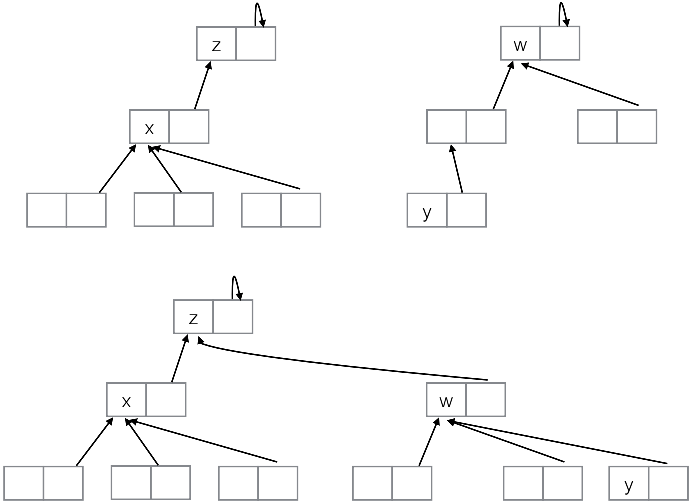

If $z \to \textbf{\textit{rank}} \geq w \to \textbf{\textit{rank}}$

If $z \to \textbf{\textit{rank}} = w \to \textbf{\textit{rank}}$,
then $(z \to \textbf{\textit{rank}})++$

Properties of Ranks

1. If $\boldsymbol{x}$ not a root, then $(\boldsymbol{x} \to \boldsymbol{rank}) < (\boldsymbol{x} \to \boldsymbol{parent} \to \boldsymbol{rank})$
2. When doing path compression, if parent of $\boldsymbol{x}$ changes, new parent has rank strictly larger than old parent
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When x first gets rank r, must be because x had rank $r - 1$ (and was root), unioned with another set with root z of rank $r - 1$.

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Proof of Property 4.

Induction. Base case: $r = 0$. ✓

Inductive case: Suppose true for $r - 1$.

When x first gets rank r, must be because x had rank $r - 1$ (and was root), unioned with another set with root z of rank $r - 1$.

$\Longrightarrow$ By induction, at least 2^{r-1} nodes in each tree

Properties of Ranks

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$\implies$ By induction, at least 2^{r-1} nodes in each tree

$\implies$ At least $2^{r-1} + 2^{r-1} = 2^r$ nodes in combined tree. □

Nodes of rank r

Lemma

There are at most $n/2^r$ nodes of rank at least r.

Proof.

Let x node of rank at least r. Let S_x be descendants of x when it first got rank r.
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So at most $2m$ Finds, want to bound total # parent pointers followed.

- At most one parent pointer to root per Find $\implies$ at most $O(m)$ parent pointers to roots.
- So only need to worry about parent pointers to non-roots.

Main Result II: Buckets

Put elements in buckets according to rank (only in analysis).

Notation: $2 \uparrow i$ denote a tower of i 2's

- $2 \uparrow 1 = 2, \quad 2 \uparrow 2 = 2^2 = 4, \quad 2 \uparrow 3 = 2^{2^2} = 2^4 = 16, \quad 2 \uparrow 4 = 2^{2^{2^2}} = 2^{16} = 65536$
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$\boldsymbol{B(i)}$ (Bucket $\boldsymbol{i}$): All elements of rank at least $\mathbf{2 \uparrow (i - 1)}$, at most $\mathbf{(2 \uparrow i) - 1}$

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$B(i)$ (Bucket i): All elements of rank at least $2 \uparrow (i-1)$, at most $(2 \uparrow i) - 1$

- Bucket 0: nodes with rank 0
- Bucket 1: rank at least 1, at most 1
- Bucket 2: rank at least 2, at most 3
- Bucket 3: rank at least 4, at most 15
- Bucket 4: rank at least 16, at most 65535
- At most $\log^* n$ buckets.

From Lemma: at most $n/(2^{2\uparrow(i-1)}) = n/(2 \uparrow i)$ elements in bucket i.

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$$\sum_x \alpha(x) = \sum_{i=0}^{O(\log^* n)} \sum_{x \in B(i)} \alpha(x) \leq \sum_{i=0}^{O(\log^* n)} \sum_{x \in B(i)} (2 \uparrow i) \leq \sum_{i=0}^{O(\log^* n)} \frac{n}{2 \uparrow i} (2 \uparrow i) = O(n \log^* n)$$

$$\leq O(m \log^* n)$$