Lecture 10: Disjoint Sets / Union-Find

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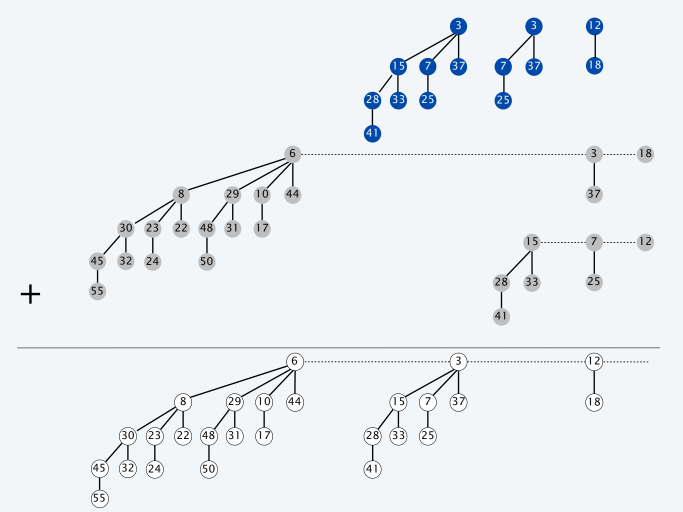
September 25, 2025
601.433/633 Introduction to Algorithms
Slides by Mike Dinitz

Meld(H_1, H_2): General Case

(Almost) just like binary addition!



Insert(H, x)

Use Meld:

- Create new heap H' with one B_0 consisting of just x
- Meld(H, H')

Correctness: Obvious

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- Worst case: $O(\log n)$ (via Meld)
- Amortized:
  - Like incrementing a binary counter!
  - If we link k trees, potential goes down by $k - 1$
  - Cost = # links plus 1 (for making new heap)
  - Amortized cost = $k + 1 + \Delta\Phi = k + 1 - (k - 1) = 2 = O(1)$

Extract-Min(H)

Use Meld again!

- $O(\log n)$ to Find-Min: one of the roots.
- Delete and return this root: tree turns into a new heap!
- Meld with original heap (minus the tree)

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Running Time:

- Worst-Case: $O(\log n)$ from creating new heap, Meld
- Amortized:
  - Potential can go up! But by at most $\log n$
  - Amortized time at most $O(\log n) + \log n = O(\log n)$

Introduction to Union-Find

Informal: Universe of elements, want to maintain *disjoint sets*.

Slightly more formally:

- Make-Set(x): create a new set containing just x (i.e., $\{x\}$)
- Union(x, y): Replace set containing x (S) and set containing y (T) with single set $S \cup T$
- Find(x): Return *representative* of set containing x

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Rules: every set has a *unique* representative.

- If x and y are in same set, Find(x) = Find(y)
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Note: disjoint (and partition) by construction!

Introduction (II)

We'll see a few ways of doing this, from efficient to very efficient.
CLRS: extremely efficient

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CLRS: extremely efficient

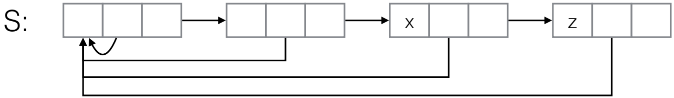
Notation and Notes:

- m operations total
- n of which are Make-Sets (so n elements)
- Assume have pointer/access to elements we care about (like last class)

First Approach: Lists

Linked list for each set.

- Representative of set is head (first element on list)
- Each element has pointer to head and to next element, so stored as triple: (element, head, next)



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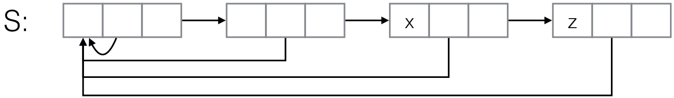

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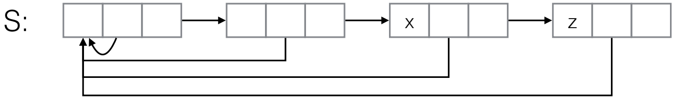
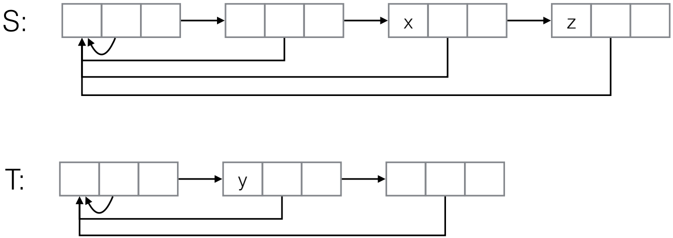


Make-Set(x):

Find(x): return $x \rightarrow$ head

Union(x, y)



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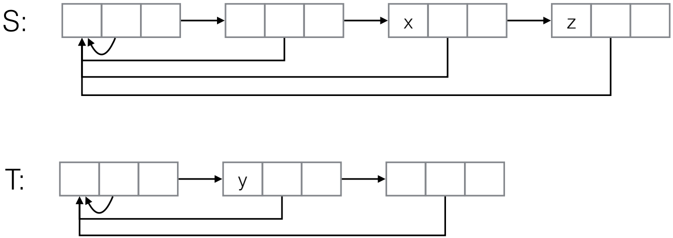

Obvious approach:

- Walk down S to final element z (starting from x)
- Set $z \rightarrow$ next = $y \rightarrow$ head
- Walk down T, set every elements head pointer to $x \rightarrow$ head

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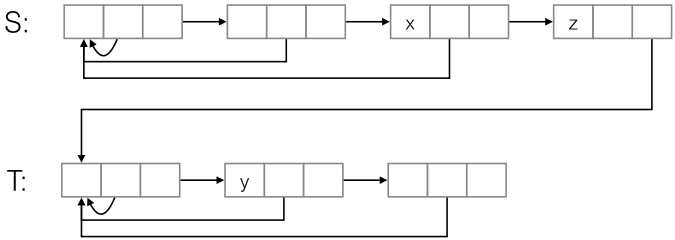

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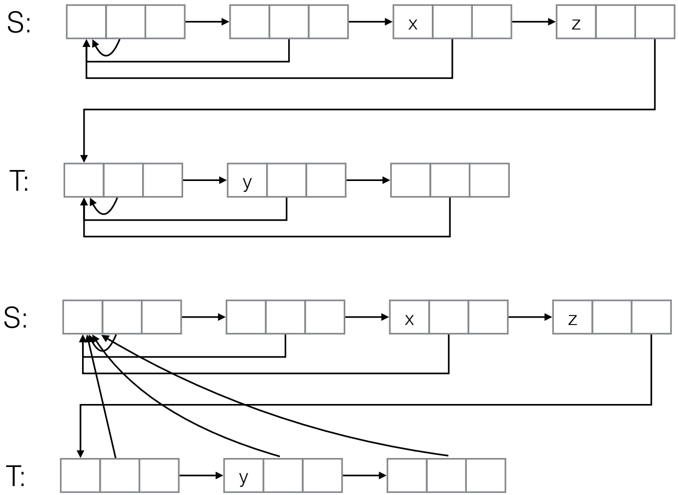

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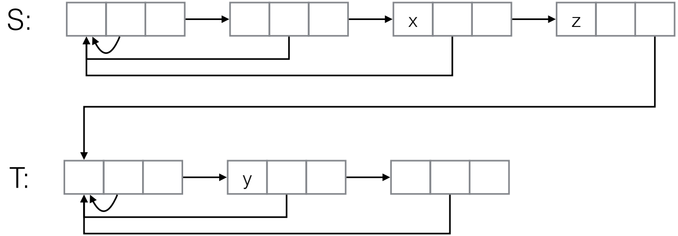
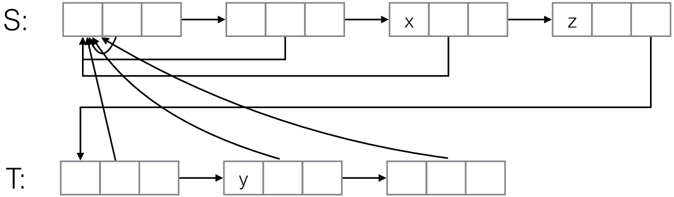
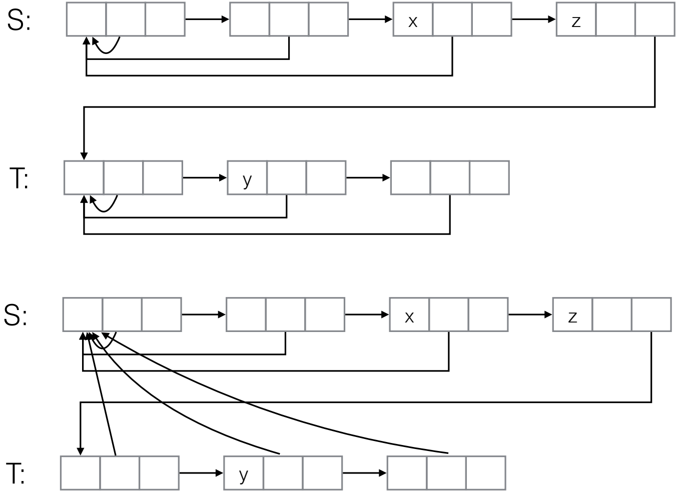




Running time:

Union(x, y)



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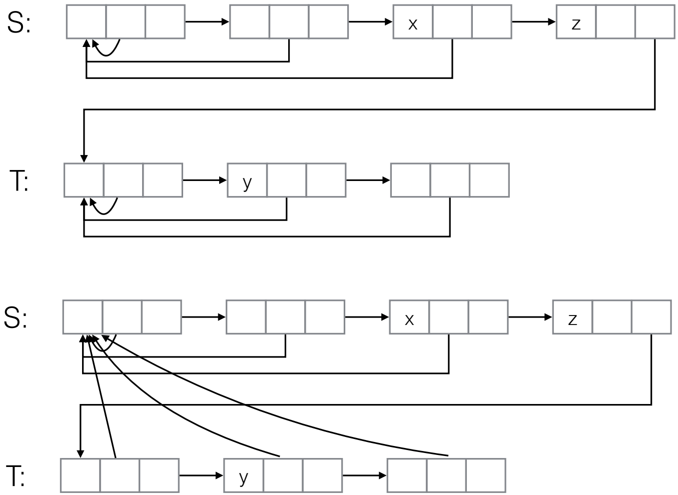

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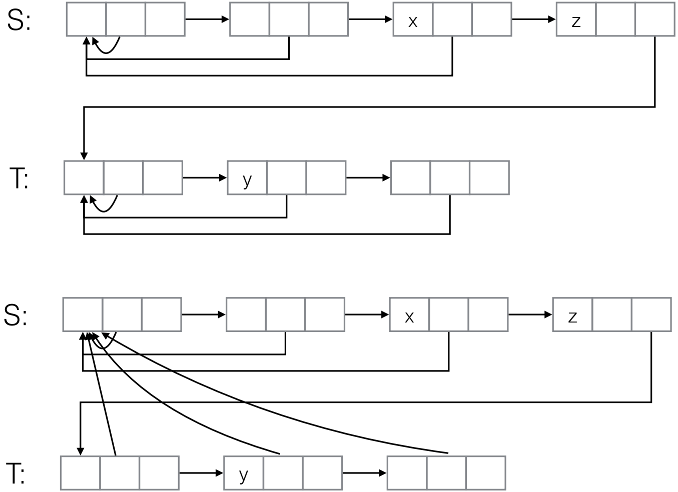

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Since $|S|, |T|$ could be $\Theta(n)$, can only say $O(n)$ for Unions

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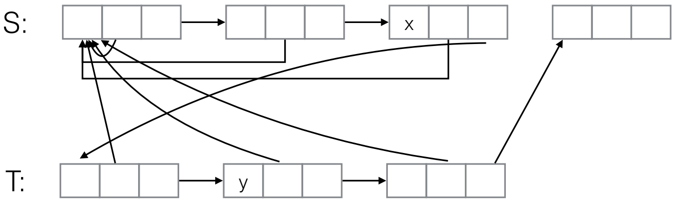

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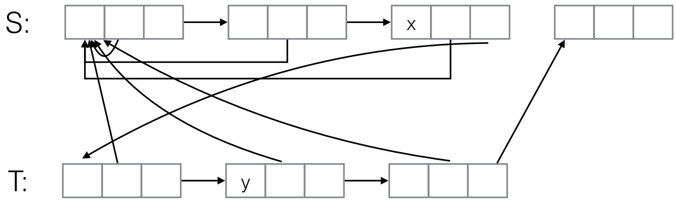

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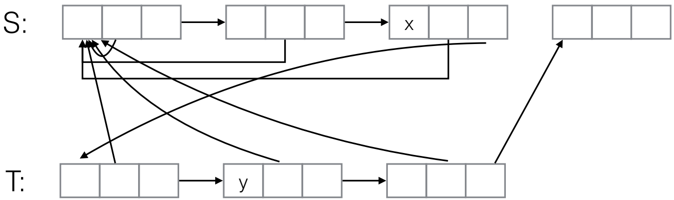

Running time: $O(|T|)$

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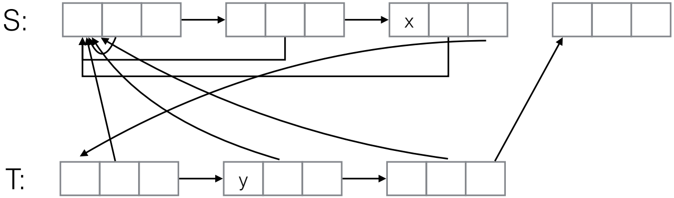

Running time: $O(|T|)$

- Still can't say anything better than $O(n)$

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Theorem

The amortized cost of Find and Union is $O(1)$, and the amortized cost of Make-Set is $O(\log n)$.

Corollary

The total running time is $O(m + n \log n)$.

Amortized Analysis of List Algorithm

Banking/accounting argument: bank for every element

- When an element is created (via Make-Set), add $\log n$ tokens to its bank
- Find does not affect banks
- When doing Union, take token from bank of each element in smaller set.

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- Can only happen at most $\log n$ times. □

Amortized Analysis of List Algorithm (cont'd)

Make-Set:

- True cost: $\boldsymbol{O(1)}$
- Change in banks: $\boldsymbol{\log n}$

$\Longrightarrow$ Amortized cost: $\boldsymbol{O(1) + O(\log n) = O(\log n)}$

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Union:

- True cost: $\min(|S|, |T|)$
- Change in banks: $-\min(|S|, |T|)$

$\Longrightarrow$ Amortized cost: $\min(|S|, |T|) - \min(|S|, |T|) = 0 = O(1)$.

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Starting idea: want to make Unions faster, willing to make Finds a little slower.

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Idea 2: *Union By Rank*

- Size of set was important for lists, less important for trees.
- Choose which set to splice into which by *rank* of trees (related to height)

Main Result

Theorem

When using Path Compression and Union By Rank, total time at most $O(m \log^* n)$.

$\log^*$: iterated $\log_2$.

- $\log^* n = \#$ times apply $\log_2$ until get to ≤ 1

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Stronger theorem: total time at most $O(m \cdot \alpha(m, n))$.

- $\alpha(m, n)$: inverse Ackermann function. Grows even slower than $\log^*$.
- See CLRS for details

Formal Procedures: Make-Set and Find

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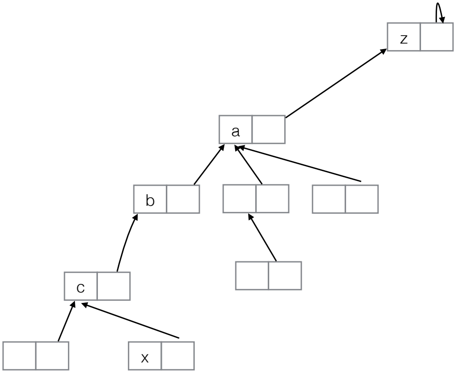
Running time of Find: depth of x (distance to root)

Find example



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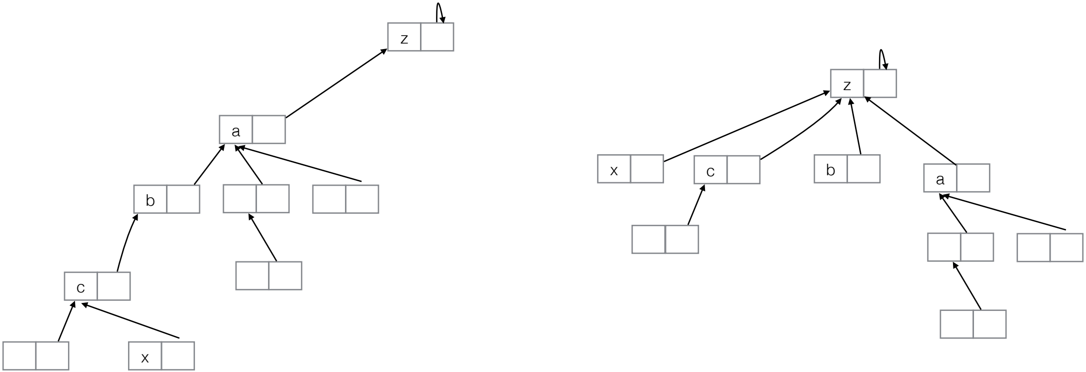

Formal Procedure: Union

Link($\boldsymbol{r_1}, \boldsymbol{r_2}$): Only applied to root nodes

- If $\boldsymbol{r_1} \rightarrow \boldsymbol{rank} > \boldsymbol{r_2} \rightarrow \boldsymbol{rank}$, set $\boldsymbol{r_2} \rightarrow \boldsymbol{parent} = \boldsymbol{r_1}$
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Union($\boldsymbol{x}, \boldsymbol{y}$): Link(Find($\boldsymbol{x}$), Find($\boldsymbol{y}$))

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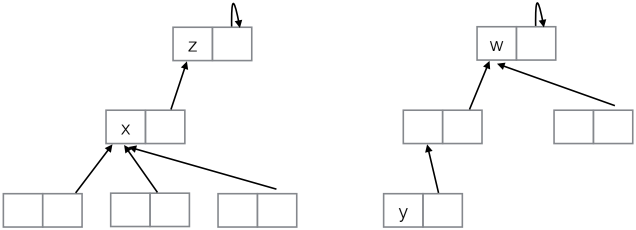
Running time of Link: $O(1)$

Union(x, y): Link(Find(x), Find(y))

- Running time: depth(x) + depth(y)

Union example



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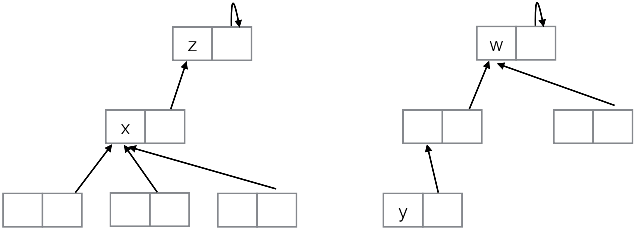

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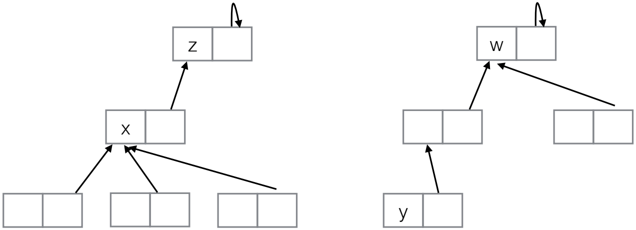

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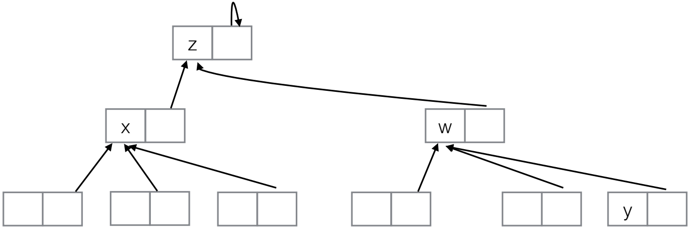

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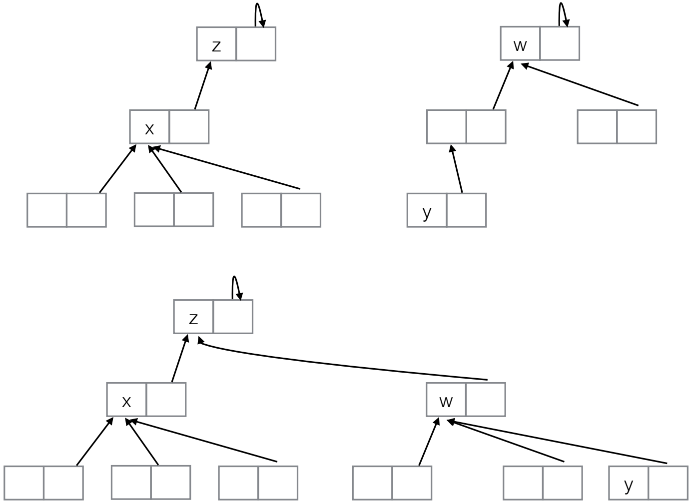

If $z \rightarrow rank \geq w \rightarrow rank$

If $z \rightarrow rank = w \rightarrow rank$,
then $(z \rightarrow rank)++$

Properties of Ranks

1. If $\boldsymbol{x}$ not a root, then $(\boldsymbol{x} \to \boldsymbol{rank}) < (\boldsymbol{x} \to \boldsymbol{parent} \to \boldsymbol{rank})$
2. When doing path compression, if parent of $\boldsymbol{x}$ changes, new parent has rank strictly larger than old parent
3. $\boldsymbol{x} \to \boldsymbol{rank}$ can change only if $\boldsymbol{x}$ a root, and once $\boldsymbol{x}$ is a non-root it never becomes a root again.

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Nodes of rank r

Lemma

There are at most $n/2^r$ *nodes of rank at least* r.

Proof.

Let x node of rank at least r. Let S_x be descendants of x when it first got rank r.
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So at most $2m$ Finds, want to bound total # parent pointers followed.

- At most one parent pointer to root per Find $\implies$ at most $O(m)$ parent pointers to roots.
- So only need to worry about parent pointers to non-roots.

Main Result II: Buckets

Put elements in buckets according to rank (only in analysis).

Notation: $\mathbf{2} \uparrow \boldsymbol{i}$ denote a tower of $\boldsymbol{i}$ $\mathbf{2}$'s

- $\mathbf{2 \uparrow 1 = 2}, \quad \mathbf{2 \uparrow 2 = 2^2 = 4}, \quad \mathbf{2 \uparrow 3 = 2^{2^2} = 2^4 = 16}, \quad \mathbf{2 \uparrow 4 = 2^{2^{2^2}} = 2^{16} = 65536}$
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From Lemma: at most $n/(2^{2\uparrow(i-1)}) = n/(2 \uparrow i)$ elements in bucket i.

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$$\sum_x \alpha(x) = \sum_{i=0}^{O(\log^* n)} \sum_{x\in B(i)} \alpha(x) \leq \sum_{i=0}^{O(\log^* n)} \sum_{x\in B(i)} (2\uparrow i) \leq \sum_{i=0}^{O(\log^* n)} \frac{n}{2\uparrow i}(2\uparrow i) = O(n\log^* n)$$

$$\leq O(m\log^* n)$$