Lecture 11: Universal and Perfect Hashing

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601.433/633 Introduction to Algorithms
Slides by Michael Dinitz

Introduction

Another approach to dictionaries (insert, lookup, delete): hashing

- Can improve operations to $\boldsymbol{O(1)}$, but with many caveats!

Should have seen some discussion of hashing in data structures. Also in CLRS.

- Separate chaining vs. open addressing

Today: discussion of caveats, more advanced versions of hashing (universal and perfect)

Hashing Basics

- Keys from universe $\boldsymbol{U}$ (think very large)
- Set $\boldsymbol{S} \subseteq \boldsymbol{U}$ of keys we actually care about (think relatively small). $|\boldsymbol{S}| = \boldsymbol{N}$.
- Hash table $\boldsymbol{A}$ (array) of size $\boldsymbol{M}$.
- Hash function $\boldsymbol{h} : \boldsymbol{U} \to [\boldsymbol{M}]$
  - $[\boldsymbol{M}] = \{1, 2, \ldots, \boldsymbol{M}\}$
- Idea: store $\boldsymbol{x}$ in $\boldsymbol{A}[\boldsymbol{h}(\boldsymbol{x})]$

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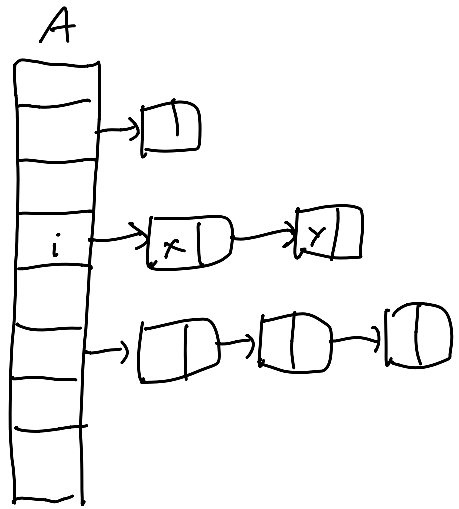
One more component: *collision resolution*

- Today: *separate chaining*
- $\boldsymbol{A}[\boldsymbol{i}]$ is a linked list containing all $\boldsymbol{x}$ inserted where $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{i}$.



Dictionary Operations

Lookup($\boldsymbol{x}$): Walk down the list at $\boldsymbol{A}[\boldsymbol{h}(\boldsymbol{x})]$ until we find $\boldsymbol{x}$ (or walk to the end of the list)

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- h fast to compute.

Bad News

Theorem

For any hash function h, *if* $|U| \geq (N-1)M+1$, *then there exists a set* S *of* N *elements that all hash to the same location.*

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Proof.

Pigeonhole principle / contradiction / contrapositive. □

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- Option 1: don't worry about it, hope world isn't adversarial.
- Option 2: Randomness! *Random function* $h : U \to [M]$
  - For each $x \in U$, choose $y \in [M]$ uniformly at random and set $h(x) = y$.
  - Hopefully good behavior in expectation.

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  - Hopefully good behavior in expectation.
  - Problem: How can we store/remember/create h?

Universal Hashing

Definition

A probability distribution H over hash functions $\{h : U \to [M]\}$ is *universal* if

$$\Pr_{h \sim H}[h(x) = h(y)] \leq 1/M$$

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If $\boldsymbol{H}$ is universal, then for every set $\boldsymbol{S} \subseteq \boldsymbol{U}$ with $|\boldsymbol{S}| = \boldsymbol{N}$ and for every $\boldsymbol{x} \in \boldsymbol{U}$, the expected number of collisions (when we draw $\boldsymbol{h}$ from $\boldsymbol{H}$) between $\boldsymbol{x}$ and elements of $\boldsymbol{S}$ is at most $\boldsymbol{N}/\boldsymbol{M}$.

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Theorem

If H is universal, then for every set $S \subseteq U$ with $|S| = N$ and for every $x \in U$, the expected number of collisions (when we draw h from H) between x and elements of S is at most N/M.

So Lookup(x) and Delete(x) have expected time $O(N/M)$.
$\implies$ If $M = \Omega(N)$, operations in $O(1)$ time!

Main Proof

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Proof.

Let $C_{xy} = \begin{cases} 1 & \text{if } h(x) = h(y) \\ 0 & \text{otherwise} \end{cases}$

$$\implies E[C_{xy}] = \Pr_{h \sim H}[h(x) = h(y)] \leq 1/M$$

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Number of collisions between x and S is exactly $\sum_{y \in S} C_{xy}$

$$\implies E\left[\sum_{y \in S} C_{xy}\right] = \sum_{y \in S} E\left[C_{xy}\right] \leq \sum_{y \in S} \frac{1}{M} = N/M$$

□

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If $\boldsymbol{H}$ is universal, then for any sequence of $\boldsymbol{L}$ insert, lookup, and delete operations in which there are at most $\boldsymbol{O(M)}$ elements in the system at any time, the expected total cost of the whole sequence is only $\boldsymbol{O(L)}$ (assuming $\boldsymbol{h}$ takes constant time to compute).

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So universal distributions are great. Can we construct them?

Universal Hash Families

Definition

If $\boldsymbol{H}$ is universal and is a uniform distribution over a set of functions $\{\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots\}$, then that set is called a *universal hash family*.

Often use $\boldsymbol{H}$ to refer to both set of functions and uniform distribution over it.

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Construction: $H = \{0,1\}^{b \times u}$, i.e., H is all $b \times u$ binary matrices

- Each $h \in H$ is a (linear) function from U to $[M]$:
  $h(x) = hx \in \{0,1\}^b$ (all operations mod 2)

$$\underset{\text{h}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}} \underset{\text{x}}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}} = \underset{\text{h(x)}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

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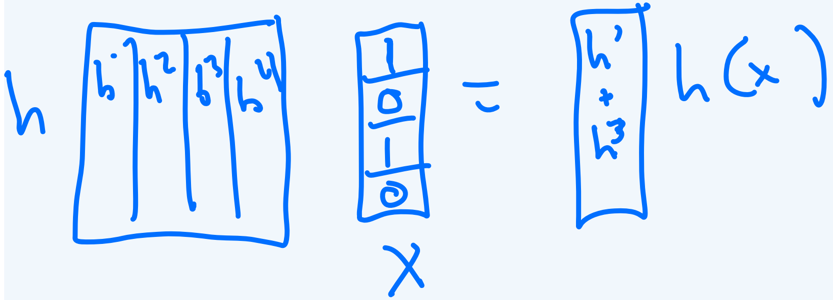

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- $h(x) = h'(x)$ already fixed.

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- $h(x) = h'(x)$ already fixed.
- If $h(y) = h(x)$, then h^i must equal $h(x) - h'(y)$
  $(h(y) = h(x) \Rightarrow \sum_{j:y_j=1} h^j = \sum_{j:x_j=1} h^j \Rightarrow h^i = \sum_{j:x_j=1} h^j - \sum_{j:y_j=1} h'^j \Rightarrow h(x) - h'(y))$

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- b rows in h, so happens with probability exactly $1/2^b = 1/M$ □

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Suppose you know $\boldsymbol{S}$, never changes.

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Can we do better with hashing? Yes, through universal hashing!

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Proof.

Fix $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S}$ with $\boldsymbol{x} \neq \boldsymbol{y}$.
$\mathbf{Pr}_{\boldsymbol{h} \sim \boldsymbol{H}}[\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{y})] \leq \mathbf{1}/\boldsymbol{M} = \mathbf{1}/\boldsymbol{N}^2$ by universality.

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$\mathbf{Pr}_{h \sim H}[h(x) = h(y)] \leq 1/M = 1/N^2$ by universality.

$$\mathbf{Pr}_{h \sim H}[\exists \text{ collision in } S] \leq \sum_{\substack{x,y \in S \\ x \neq y}} \mathbf{Pr}_{h \sim H}[h(x) = h(y)] \leq \sum_{\substack{x,y \in S \\ x \neq y}} \frac{1}{N^2}$$

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So keep sampling $h \sim H$ until get one with no collisions!

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For each $i \in [M]$, let $S_i = \{x \in S : h(x) = i\}$ and let $n_i = |S_i|$

- Use another hash table for S_i!
- Use Method 1: $O(n_i^2)$-size perfect hashing of S_i.
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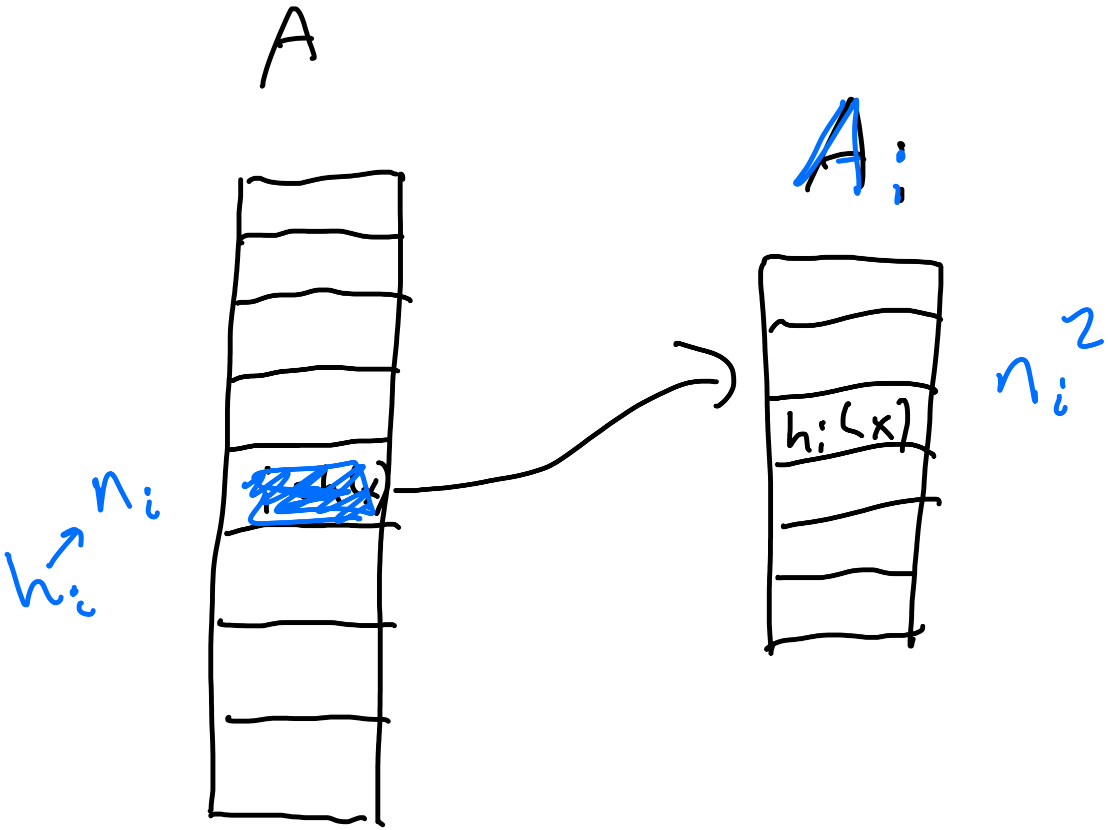
Lookup(x): Look in $A_{h(x)}[h_{h(x)}(x)]$

Picture



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Theorem

Let H be universal onto a table of size N. Then

$$\Pr_{h \sim H}\left[\sum_{i=1}^{N} n_i^2 > 4N\right] < 1/2.$$

So like with method 1: keep drawing $h \sim H$ until $\sum_{i=1}^{N} n_i^2 \leq 4N$

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Prove that $E\left[\sum_{i=1}^{N} n_i^2\right] \leq 2N$.

- Implies theorem by Markov's inequality
  - $\Pr[X > 2E[X]] \leq 1/2$ for nonnegative random variables X.

Proof

Observation: $\sum_{i=1}^{N} n_i^2$ is exactly number of *ordered* pairs that collide, including self-collisions

- Example: If $S_i = \{a, b, c\}$ then $n_i^2 = 9$. Ordered colliding pairs: $(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)$

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Proof

Observation: $\sum_{i=1}^{N} n_i^2$ is exactly number of *ordered* pairs that collide, including self-collisions

- Example: If $S_i = \{a, b, c\}$ then $n_i^2 = 9$. Ordered colliding pairs: $(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)$

Let $C_{xy} = \begin{cases} 1 & \text{if } h(x) = h(y) \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} E\left[\sum_{i=1}^{N} n_i^2\right] &= E\left[\sum_{x \in S} \sum_{y \in S} C_{xy}\right] \\ &= N + \sum_{x \in S} \sum_{y \in S : y \neq x} E\left[C_{xy}\right] && \text{(linearity of expectations)} \\ &\leq N + \frac{N(N-1)}{M} && \text{(definition of universal)} \\ &< 2N && (\text{since } M = N) \end{aligned}$$