Lecture 11: Universal and Perfect Hashing

Michael Dinitz

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601.433/633 Introduction to Algorithms

Introduction

Another approach to dictionaries (insert, lookup, delete): hashing

- Can improve operations to $\boldsymbol{O(1)}$, but with many caveats!

Should have seen some discussion of hashing in data structures. Also in CLRS.

- Separate chaining vs. open addressing

Today: discussion of caveats, more advanced versions of hashing (universal and perfect)

Hashing Basics

- Keys from universe U (think very large)
- Set $S \subseteq U$ of keys we actually care about (think relatively small). $|S| = N$.
- Hash table A (array) of size M.
- Hash function $h : U \to [M]$
  - $[M] = \{1, 2, \ldots, M\}$
- Idea: store x in $A[h(x)]$

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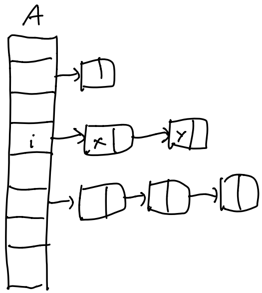
One more component: *collision resolution*

- Today: *separate chaining*
- $A[i]$ is a linked list containing all x inserted where $h(x) = i$.



Dictionary Operations

Lookup(x): Walk down the list at $A[h(x)]$ until we find x (or walk to the end of the list)

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- Small M. Ideally, $M = O(N)$.
- h fast to compute.

Bad News

Theorem

For any hash function $\boldsymbol{h}$, if $|\boldsymbol{U}| \geq (\boldsymbol{N}-1)\boldsymbol{M}+1$, then there exists a set $\boldsymbol{S}$ of $\boldsymbol{N}$ elements that all hash to the same location.

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Pigeonhole principle / contradiction / contrapositive. $\square$

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- Option 2: Randomness! *Random function* $\boldsymbol{h}: \boldsymbol{U} \to [\boldsymbol{M}]$
  - For each $\boldsymbol{x} \in \boldsymbol{U}$, choose $\boldsymbol{y} \in [\boldsymbol{M}]$ uniformly at random and set $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{y}$.
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  - Hopefully good behavior in expectation.
  - Problem: How can we store/remember/create $\boldsymbol{h}$?

Universal Hashing

Definition

A probability distribution H over hash functions $\{h : U \to [M]\}$ is *universal* if

$$\Pr_{h \sim H}[h(x) = h(y)] \le 1/M$$

for all $x, y \in U$ with $x \neq y$.

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So Lookup(x) and Delete(x) have expected time $O(N/M)$.
$\implies$ If $M = \Omega(N)$, operations in $O(1)$ time!

Main Proof

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Proof.

Let $C_{xy} = \begin{cases} 1 & \text{if } h(x) = h(y) \\ 0 & \text{otherwise} \end{cases}$

$$\implies E[C_{xy}] = \Pr_{h \sim H}[h(x) = h(y)] \leq 1/M$$

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Number of collisions between x and S is exactly $\sum_{y \in S} C_{xy}$

$$\implies E\left[\sum_{y \in S} C_{xy}\right] = \sum_{y \in S} E\left[C_{xy}\right] \leq \sum_{y \in S} \frac{1}{M} = N/M \qquad \square$$

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If H is universal, then for any sequence of L insert, lookup, and delete operations in which there are at most $O(M)$ elements in the system at any time, the expected total cost of the whole sequence is only $O(L)$ (assuming h takes constant time to compute).

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So universal distributions are great. Can we construct them?

Universal Hash Families

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If $\boldsymbol{H}$ is universal and is a uniform distribution over a set of functions $\{\boldsymbol{h_1}, \boldsymbol{h_2}, \ldots\}$, then that set is called a *universal hash family*.

Often use $\boldsymbol{H}$ to refer to both set of functions and uniform distribution over it.

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- $M = 2^b$, so an index to A is an element of $\{0,1\}^b$

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$$\begin{array}{cccc} \text{h} & \text{x} & & \text{h(x)} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} & = & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{array}$$

Construction: $H = \{0,1\}^{b \times u}$, i.e., H is all $b \times u$ binary matrices

- Each $h \in H$ is a (linear) function from U to $[M]$: $h(x) = hx \in \{0,1\}^b$ (all operations mod 2)

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Theorem

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Draw all entries of h *except* for h^i. Let $h' = h$ with h^i all $\mathbf{0}$'s

- $h(x) = h'(x)$ already fixed.

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Proof.

Matrix multiplication: $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{h}\boldsymbol{x} = \sum_{\boldsymbol{i}:\boldsymbol{x}_{\boldsymbol{i}}=\mathbf{1}} \boldsymbol{h}^{\boldsymbol{i}}$ (where $\boldsymbol{h}^{\boldsymbol{i}}$ is $\boldsymbol{i}$'th column of $\boldsymbol{h}$).

Since $\boldsymbol{x} \neq \boldsymbol{y}$, there is $\boldsymbol{i}$ s.t. $\boldsymbol{x}_{\boldsymbol{i}} \neq \boldsymbol{y}_{\boldsymbol{i}}$. WLOG, $\boldsymbol{x}_{\boldsymbol{i}} = \mathbf{0}$ and $\boldsymbol{y}_{\boldsymbol{i}} = \mathbf{1}$.

Draw all entries of $\boldsymbol{h}$ *except* for $\boldsymbol{h}^{\boldsymbol{i}}$. Let $\boldsymbol{h}' = \boldsymbol{h}$ with $\boldsymbol{h}^{\boldsymbol{i}}$ all $\mathbf{0}$'s

- $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{h}'(\boldsymbol{x})$ already fixed.
- If $\boldsymbol{h}(\boldsymbol{y}) = \boldsymbol{h}(\boldsymbol{x})$, then $\boldsymbol{h}^{\boldsymbol{i}}$ must equal $\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{h}'(\boldsymbol{y})$

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- $h(x) = h'(x)$ already fixed.
- If $h(y) = h(x)$, then h^i must equal $h(x) - h'(y)$
- Happens with probability exactly $1/2^b = 1/M$ $\square$

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Can we do better with hashing? Yes, through universal hashing!

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$$\Pr_{h \sim H}[\exists \text{ collision in } S] \leq \sum_{\substack{x,y \in S \\ x \neq y}} \Pr_{h \sim H}[h(x) = h(y)] \leq \sum_{\substack{x,y \in S \\ x \neq y}} \frac{1}{N^2}$$
$$= \binom{N}{2} \frac{1}{N^2} = \frac{N(N-1)}{2} \frac{1}{N^2} \leq \frac{1}{2}$$

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So keep sampling $h \sim H$ until get one with no collisions!

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- Use another hash table for S_i!
- Use Method 1: $O(n_i^2)$-size perfect hashing of S_i.
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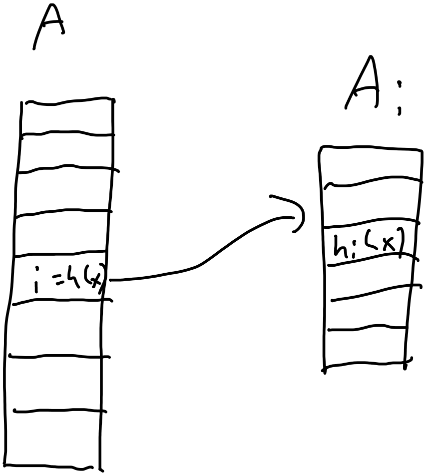
Lookup(x): Look in $A_{h(x)}[h_{h(x)}(x)]$

Picture



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Theorem

Let H be universal onto a table of size N. Then

$$\Pr_{h \sim H}\left[\sum_{i=1}^{N} n_i^2 > 4N\right] < 1/2.$$

So like with method 1: keep drawing $h \sim H$ until $\sum_{i=1}^{N} n_i^2 \leq 4N$

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Prove that $E\left[\sum_{i=1}^{N} n_i^2\right] \le 2N$.

- Implies theorem by Markov's inequality
  - $\Pr[X > 2E[X]] \le 1/2$ for nonnegative random variables X.

Proof

Observation: $\sum_{i=1}^{N} n_i^2$ is exactly number of *ordered* pairs that collide, including self-collisions

- Example: If $S_i = \{a, b, c\}$ then $n_i^2 = 9$. Ordered colliding pairs: $(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)$

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Let $C_{xy} = \begin{cases} 1 & \text{if } h(x) = h(y) \\ 0 & \text{otherwise} \end{cases}$

$$
\begin{aligned}
E\left[\sum_{i=1}^{N} n_i^2\right] &= E\left[\sum_{x \in S}\sum_{y \in S} C_{xy}\right] \\
&= N + \sum_{x \in S}\sum_{y \in S: y \neq x} E\left[C_{xy}\right] && \text{(linearity of expectations)} \\
&\leq N + \frac{N(N-1)}{M} && \text{(definition of universal)} \\
&< 2N && \text{(since } M = N\text{)}
\end{aligned}
$$