Lecture 13: Dynamic Programming II

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October 7, 2025

601.433/633 Introduction to Algorithms

Introduction

Today: two more examples of dynamic programming

- *Longest Common Subsequence* (strings)
- *Optimal Binary Search Tree* (trees)

Important problems, but really: more examples of dynamic programming

Both in CLRS (unlike Weighted Interval Scheduling)

Longest Common Subsequence

Definitions

String: Sequence of elements of some *alphabet* ($\{0, 1\}$, or $\{A - Z\} \cup \{a - z\}$, etc.)

Definition: A sequence $Z = (z_1, \ldots, z_k)$ is a *subsequence* of $X = (x_1, \ldots, x_m)$ if there exists a strictly increasing sequence $(i_1, i_2, \ldots, i_k)$ such that $x_{i_j} = z_j$ for all $j \in \{1, 2, \ldots, k\}$.

Example: (B, C, D, B) is a subsequence of (A, B, C, B, D, A, B)

- Allowed to skip positions, unlike substring!

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Definition: In *Longest Common Subsequence* problem (LCS) we are given two strings $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots y_n)$. Need to find the longest Z which is a subsequence of both X and Y.

Subproblems

First and most important step of dynamic programming: define subproblems!

- Not obvious: $\boldsymbol{X}$ and $\boldsymbol{Y}$ might not even be same length!

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Prefixes of strings

- $\boldsymbol{X_i = (x_1, x_2, \ldots, x_i)}$ (so $\boldsymbol{X = X_m}$)
- $\boldsymbol{Y_j = (y_1, y_2, \ldots, y_j)}$ (so $\boldsymbol{Y = Y_n}$)

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Definition: Let $OPT(i, j)$ be longest common subsequence of X_i and Y_j

So looking for optimal solution $OPT = OPT(m, n)$

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Two-dimensional table!

Optimal Substructure

Second step of dynamic programming: prove optimal substructure

- Relationship between subproblems: show that solution to subproblem can be found from solutions to smaller subproblems

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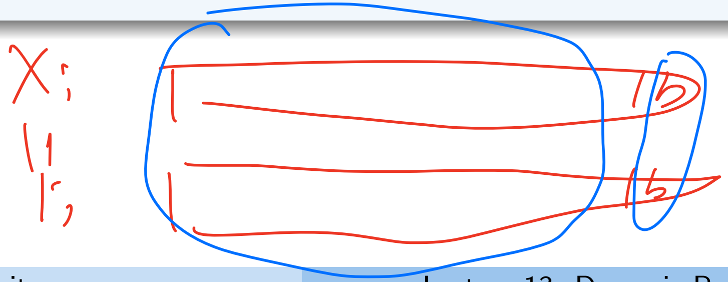

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1. *If* $\boldsymbol{x_i = y_j}$*: then* $\boldsymbol{z_k = x_i = y_j}$ *and* $\boldsymbol{Z_{k-1} = OPT(i-1, j-1)}$
2. *If* $\boldsymbol{x_i \neq y_j}$ *and* $\boldsymbol{z_k \neq x_i}$*:*

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Optimal Substructure: Proof (I)

Case 1: If $x_i = y_j$, then $z_k = x_i = y_j$ and $Z_{k-1} = OPT(i-1, j-1)$

Proof Sketch.

Contradiction.

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Part 2: Suppose $Z_{k-1} \neq OPT(i-1, j-1)$.

$\implies$ $\exists W$ LCS of X_{i-1}, Y_{j-1} of length $> k-1 \implies \geq k$

$\implies$ (W, a) common subsequence of X_i, Y_j of length $> k$

- Contradiction to Z being LCS of X_i and Y_j $\square$

Optimal Substructure: Proof (II)

Case 2: If $x_i \neq y_j$ and $z_k \neq x_i$ then $Z = OPT(i-1, j)$

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$\implies Z = OPT(i-1, j)$ □

Optimal Substructure: Proof (III)

Case 3: If $x_i \neq y_j$ and $z_k \neq y_j$ then $Z = OPT(i, j-1)$

Proof.

Symmetric to Case 2. $\square$

Structure Corollary

$(OPT(i-1, j-1), x_i)$

Corollary

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$$OPT(i,j) = \begin{cases} \emptyset & \text{if } i = 0 \text{ or } j = 0, \\ OPT(i-1,j-1) \circ x_i & \text{if } i,j > 0 \text{ and } x_i = y_j \\ \text{arg}\max(|OPT(i,j-1)|, |OPT(i-1,j)|) & \text{if } i,j > 0 \text{ and } x_i \neq y_j \end{cases}$$

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Gives obvious recursive algorithm

- Can take exponential time (good exercise at home!)

Dynamic Programming!

- Top-Down: are problems getting "smaller"? What does "smaller" mean?
- Bottom-Up: two-dimensional table! What order to fill it in?

Dynamic Programming Algorithm

LCS(X,Y) {
    for($i = 0$ to m) $M[i, 0] = 0$;
    for($j = 0$ to n) $M[0, j] = 0$;
    for($i = 1$ to m) {
        for($j = 1$ to n) {
            if($x_i = y_j$)
                $M[i, j] = 1 + M[i-1, j-1]$;
            else
                $M[i, j] = \max(M[i, j-1], M[i-1, j])$;
        }
    }
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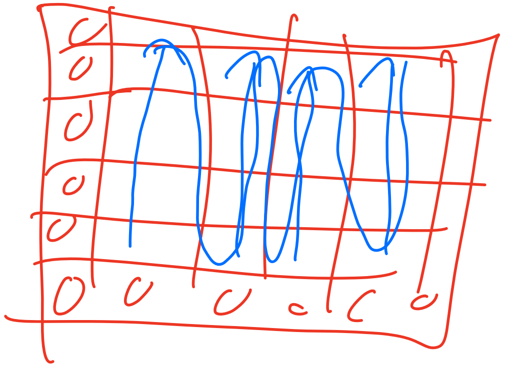

Running Time: $O(mn)$

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Inductive Step: Divide into three cases

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alg   induction   ST

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3. If $x_i \neq y_j$, then

$$\begin{aligned} M[i,j] &= \max(M[i,j-1], M[i-1,j]) && \text{(def of algorithm)} \\ &= \max(|OPT(i,j-1)|, |OPT(i-1,j)|) && \text{(induction)} \\ &= |OPT(i,j)| && \text{(structure thm/corollary)} \end{aligned}$$

Computing a Solution

Like we talked about last lecture: backtrack through dynamic programming table.

Details in CLRS 14.4

Optimal Binary Search Trees

Problem Definition

Input: probability distribution / search frequency of keys

- n distinct keys $k_1 < k_2 < \cdots < k_n$
- For each $i \in [n]$, probability p_i that we search for k_i (so $\sum_{i=1}^{n} p_i = 1$)

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Cost of searching for k_i in tree T is $depth_T(k_i) + 1$ (say depth of root $= 0$)

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Definition: $c(T) = \sum_{i=1}^{n} p_i(depth_T(k_i) + 1)$

Problem: Find search tree T minimizing cost.

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Natural approach: greedy (make highest probability key the root). Does this work?

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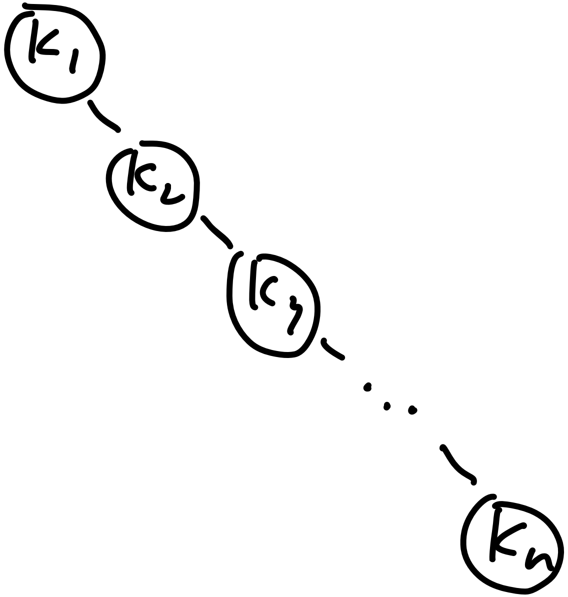

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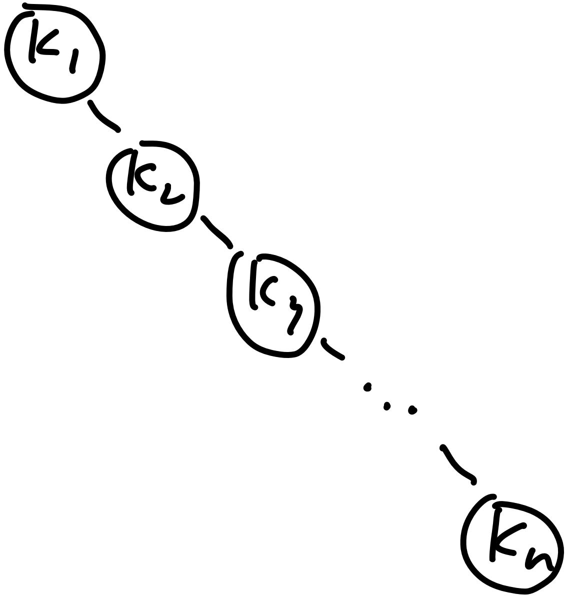

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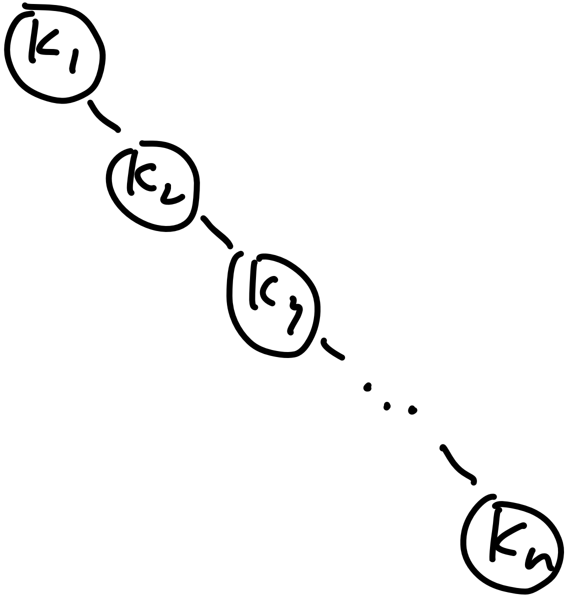
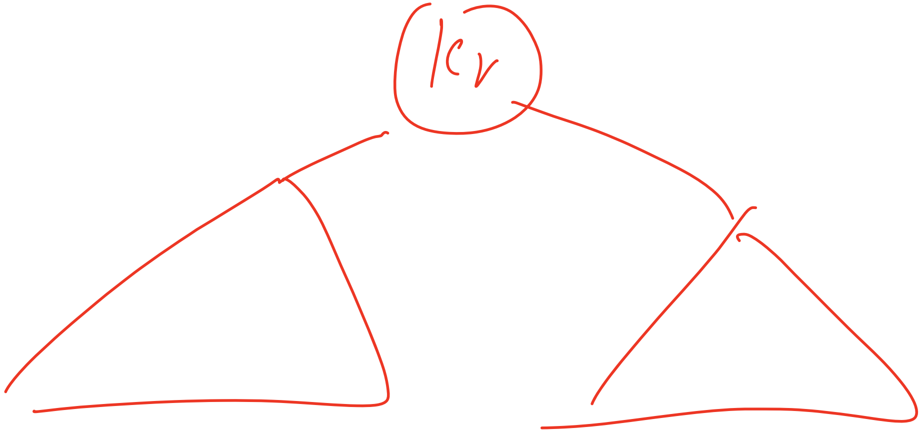


$\boldsymbol{E}[\text{cost of search in } \boldsymbol{T}] \geq \boldsymbol{\Omega(n)}$

Balanced search tree: $\boldsymbol{E}[\text{cost}] \leq \boldsymbol{O(\log n)}$

Intuition

Suppose root is $\boldsymbol{k_r}$. What does optimal tree look like?



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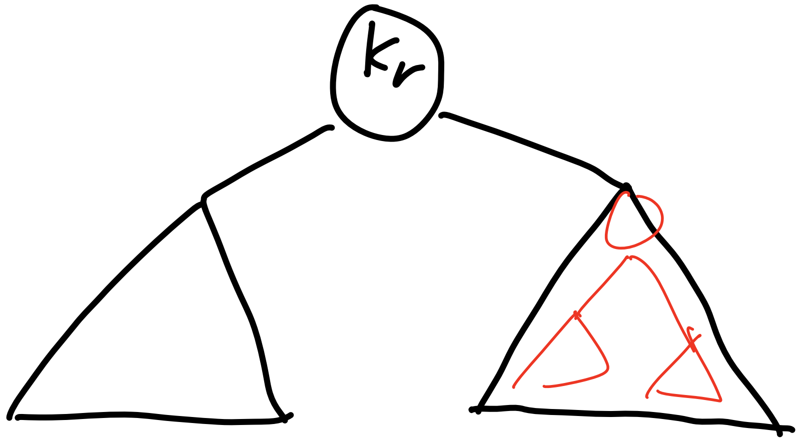

OPT for $\{k_1, \ldots, k_{r-1}\}$

OPT for $\{k_{r+1}, \ldots, k_n\}$

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Let $OPT(i,j)$ with $i \le j$ be optimal tree for keys $\{k_i, k_{i+1}, \ldots, k_j\}$: tree T minimizing $c(T) = \sum_{a=i}^{j} p_a(depth_T(k_a) + 1)$

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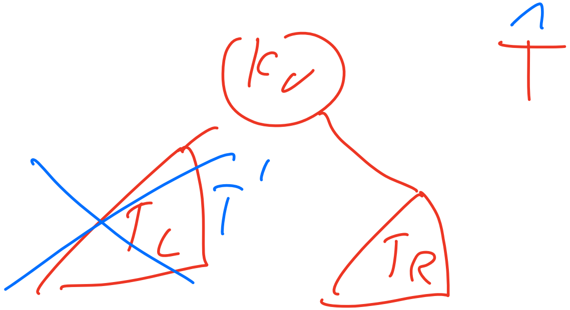
Theorem (Optimal Substructure)

Let k_r be the root of $OPT(i,j)$. Then the left subtree of $OPT(i,j)$ is $OPT(i,r-1)$, and the right subtree of $OPT(i,j)$ is $OPT(r+1,j)$.

Proof Sketch of Optimal Substructure



Definitions:

- Let $T = OPT(i,j)$, T_L its left subtree, T_R its right subtree.
- Suppose for contradiction $T_L \neq OPT(i, r-1)$, let $T' = OPT(i, r-1)$

  $\implies$ $c(T') < c(T_L)$ (def of $OPT(i, r-1)$)
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Symmetric argument works for $T_R = OPT(r+1, j)$

Cost Corollary

Corollary

$c(OPT(i,j)) = \sum_{a=i}^{j} p_a + \min_{i \le r \le j}(c(OPT(i,r-1)) + c(OPT(r+1,j)))$

Let k_r be root of $OPT(i,j)$

$$c(OPT(i,j)) = \sum_{a=i}^{j} p_a(depth_{OPT(i,j)}(k_a) + 1) \quad \text{def}$$

$$\text{ST} \rightarrow \quad = \sum_{a=i}^{r-1}(p_a(depth_{OPT(i,r-1)}(k_a) + 2)) + p_r + \sum_{a=r+1}^{j} p_a(depth_{OPT(r+1,j)}(k_a) + 2)$$

$$\text{algebra} \rightarrow \quad = \sum_{a=i}^{j} p_a + \sum_{a=i}^{r-1}(p_a(depth_{OPT(i,r-1)}(k_a) + 1)) + \sum_{a=r+1}^{j} p_a(depth_{OPT(r+1,j)}(k_a) + 1)$$

$$\text{def of cost} \rightarrow \quad = \sum_{a=i}^{j} p_a + c(OPT(i,r-1)) + c(OPT(r+1,j)).$$

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c(OPT(i,j)) &= \sum_{a=i}^{j} p_a(depth_{OPT(i,j)}(k_a) + 1) \\
&= \sum_{a=i}^{r-1}(p_a(depth_{OPT(i,r-1)}(k_a) + 2)) + p_r + \sum_{a=r+1}^{j} p_a(depth_{OPT(r+1,j)}(k_a) + 2) \\
&= \sum_{a=i}^{j} p_a + \sum_{a=i}^{r-1}(p_a(depth_{OPT(i,r-1)}(k_a) + 1)) + \sum_{a=r+1}^{j} p_a(depth_{OPT(r+1,j)}(k_a) + 1) \\
&= \sum_{a=i}^{j} p_a + c(OPT(i, r-1)) + c(OPT(r+1, j)).
\end{aligned}
$$

Same logic holds for any possible root $\implies$ take min

Algorithm

Fill in table M:

$$M[i,j] = \begin{cases} 0 & \text{if } i > j \\ \min_{i \le r \le j} \left(\sum_{a=i}^{j} p_a + M[i, r-1] + M[r+1, j] \right) & \text{if } i \le j \end{cases}$$

Algorithm

Fill in table $\boldsymbol{M}$:

$$\boldsymbol{M}[\boldsymbol{i},\boldsymbol{j}] = \begin{cases} \mathbf{0} & \text{if } \boldsymbol{i} > \boldsymbol{j} \\ \mathbf{min}_{\boldsymbol{i} \le \boldsymbol{r} \le \boldsymbol{j}} \left(\sum_{\boldsymbol{a}=\boldsymbol{i}}^{\boldsymbol{j}} \boldsymbol{p}_{\boldsymbol{a}} + \boldsymbol{M}[\boldsymbol{i}, \boldsymbol{r}-\mathbf{1}] + \boldsymbol{M}[\boldsymbol{r}+\mathbf{1}, \boldsymbol{j}] \right) & \text{if } \boldsymbol{i} \le \boldsymbol{j} \end{cases}$$

Top-Down (memoization): are problems getting smaller?

Algorithm

Fill in table $\boldsymbol{M}$:

$$\boldsymbol{M[i,j]} = \begin{cases} \mathbf{0} & \text{if } \boldsymbol{i > j} \\ \mathbf{min}_{\boldsymbol{i \le r \le j}} \left(\sum_{\boldsymbol{a=i}}^{\boldsymbol{j}} \boldsymbol{p_a} + \boldsymbol{M[i, r-1]} + \boldsymbol{M[r+1, j]} \right) & \text{if } \boldsymbol{i \le j} \end{cases}$$

Top-Down (memoization): are problems getting smaller? Yes! $\boldsymbol{j - i}$ decreases in every recursive call.

Algorithm

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Top-Down (memoization): are problems getting smaller? Yes! $j - i$ decreases in every recursive call.

Correctness. Claim $M[i,j] = c(OPT(i,j))$. Induction on $j - i$.

Algorithm

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- Base case: if $j - i < 0$ then $M[i,j] = OPT(i,j) = 0$

Algorithm

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$$M[i,j] = \begin{cases} 0 & \text{if } i > j \\ \min_{i \le r \le j} \left(\sum_{a=i}^{j} p_a + M[i, r-1] + M[r+1, j] \right) & \text{if } i \le j \end{cases}$$

Top-Down (memoization): are problems getting smaller? Yes! $j - i$ decreases in every recursive call.

Correctness. Claim $M[i,j] = c(OPT(i,j))$. Induction on $j - i$.

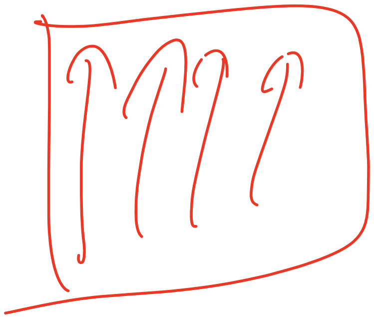
- Base case: if $j - i < 0$ then $M[i,j] = OPT(i,j) = 0$
- Inductive step:

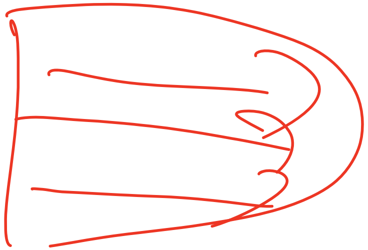
$$\begin{aligned} M[i,j] &= \min_{i \le r \le j} \left(\sum_{a=i}^{j} p_a + M[i, r-1] + M[r+1, j] \right) && \text{(alg def)} \\ &= \min_{i \le r \le j} \left(\sum_{a=i}^{j} p_a + c(OPT(i, r-1)) + c(OPT(r+1, j)) \right) && \text{(induction)} \\ &= c(OPT(i,j)) && \text{(cost corollary)} \end{aligned}$$

Algorithm: Bottom-up

What order to fill the table in?

- Obvious approach: for($\boldsymbol{i} = \mathbf{1}$ to $\boldsymbol{n} - \mathbf{1}$) for($\boldsymbol{j} = \boldsymbol{i} + \mathbf{1}$ to $\boldsymbol{n}$) Doesn't work!

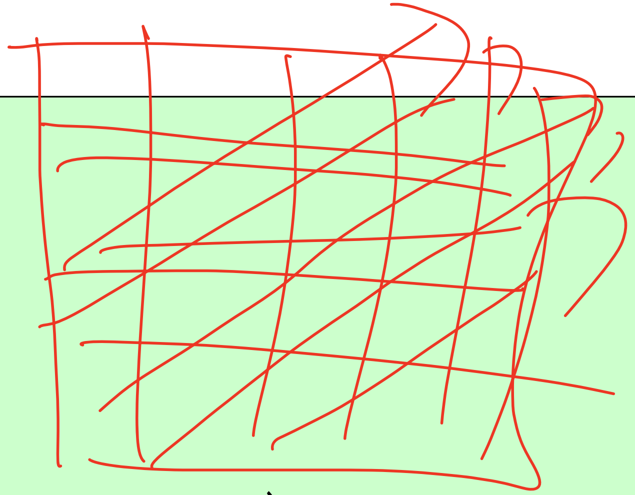




Algorithm: Bottom-up

What order to fill the table in?

- Obvious approach: for($i = 1$ to $n - 1$) for($j = i + 1$ to n) Doesn't work!
- Take hint from induction: $j - i$



OBST {
    Set $M[i, j] = 0$ for all $j > i$;
    Set $M[i, i] = p_i$ for all i
    for($\ell = 1$ to $n - 1$) {
        for($i = 1$ to $n - \ell$) {
            $j = i + \ell$
            $M[i, j] = \min_{i \le r \le j} \left(\sum_{a=i}^{j} p_a + M[i, r-1] + M[r+1, j] \right)$;
        }
    }
    return $M[1, n]$;
}

Analysis

Correctness: same as top-down

Running Time:

Analysis

Correctness: same as top-down

Running Time:

- # table entries:

Analysis

Correctness: same as top-down

Running Time:

- # table entries: $O(n^2)$

Analysis

Correctness: same as top-down

Running Time:

- # table entries: $O(n^2)$
- Time to compute table entry $M[i,j]$:

Analysis

Correctness: same as top-down

Running Time:

- # table entries: $O(n^2)$
- Time to compute table entry $M[i,j]$: $O(j - i) = O(n)$

Analysis

Correctness: same as top-down

Running Time:

- # table entries: $O(n^2)$
- Time to compute table entry $M[i,j]$: $O(j-i) = O(n)$

Total running time: $O(n^3)$

Bonus Content

Obvious Question: Robustness.

- What if given distribution is *wrong*?

Want algorithm that gives a solution with cost a function of true optimal cost, “distance” between given distribution and true distribution.

Dinitz, Im, Lavastida, Moseley, Niaparast, Vassilvitskii. *Binary Search Trees with Distributional Predictions*. NeurIPS ’24