Lecture 13: Basic Graph Algorithms

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October 8, 2024
601.433/633 Introduction to Algorithms

Introduction

Next 3-4 weeks: graphs!

- Super important abstractions, used all over the place in CS
- Most of my research is in graph algorithms (particularly when graph represents computer/communication network)
- Great course on Graph Theory in AMS

Today: review of basic graph algorithms from Data Structures, possibly one or two new

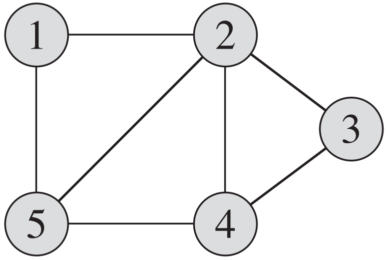
Basic Definitions

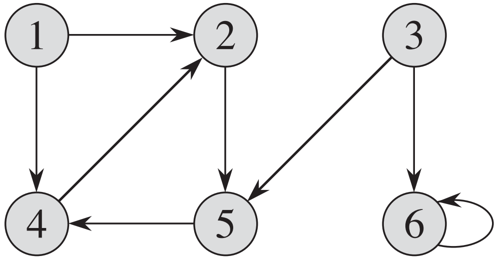
Definition

A graph $G = (V, E)$ is a pair where V is a set and $E \subseteq \binom{V}{2}$ (unordered pairs) or $E \subseteq V \times V$ (ordered pairs).

Notation:

- Elements of V are called *vertices* or *nodes*
- Elements of E are called *edges* or *arcs*.
- If $E \subseteq \binom{V}{2}$ then graph is *undirected*, if $E \subseteq V \times V$ graph is *directed*
- $|V| = n$ and $|E| = m$ (usually)
- So "size of input" $= n + m$





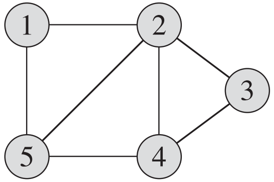
Representations

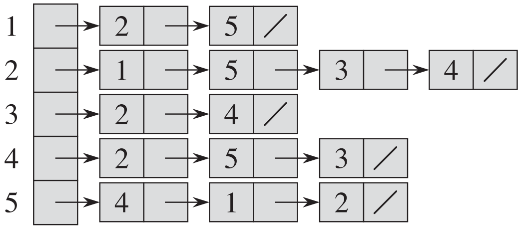
Adjacency List:

- Array A of length n
- $A[v]$ is linked list of vertices *adjacent* to v (edge from u to v)

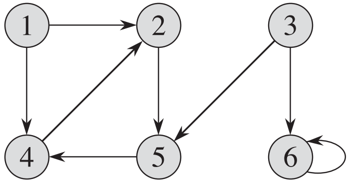
Adjacency Matrix:

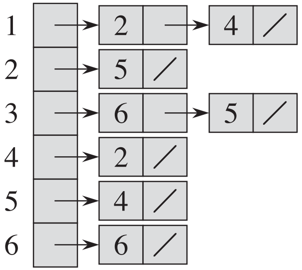
- $A \in \{0,1\}^{n \times n}$
- $A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$





$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$





$$\begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 & 1 \\ 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

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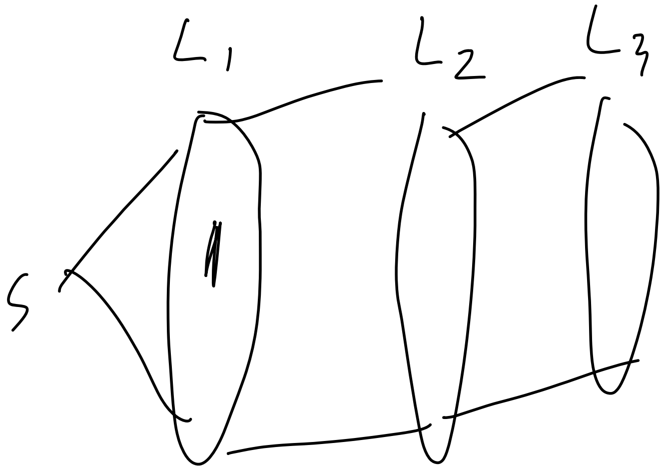
- Replace adjacency *list* with adjacency *structure*: Red-black tree, hash table, etc.
- Not traditional, doesn't gain us much, and more complicated. But better!

Breadth-First Search (BFS)

BFS Definition

Idea: explore graph in *levels* or *layers* from source $\boldsymbol{s}$



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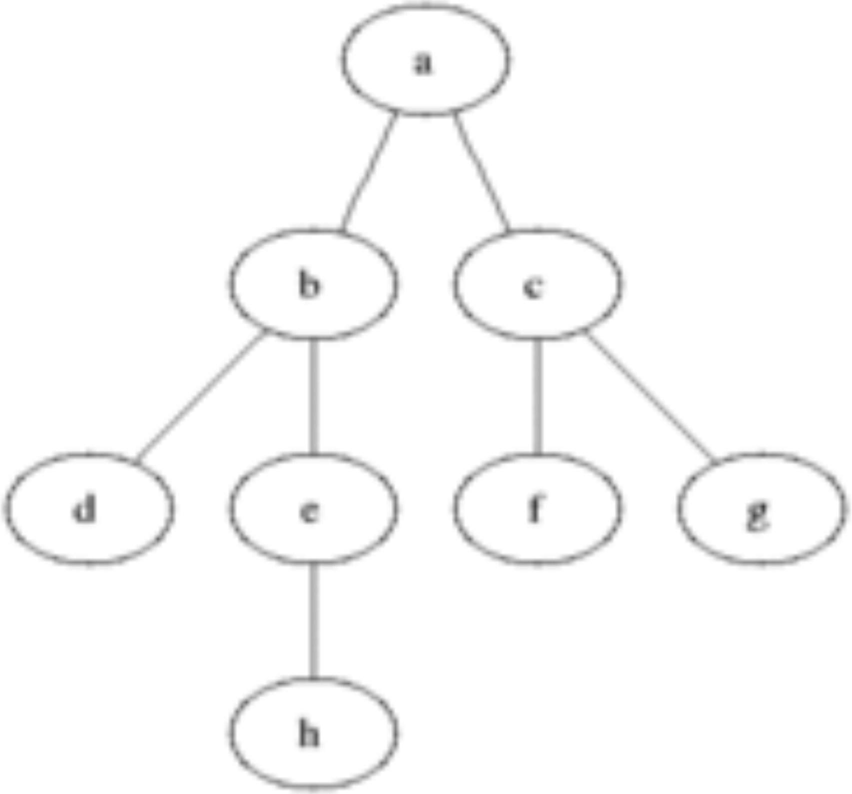

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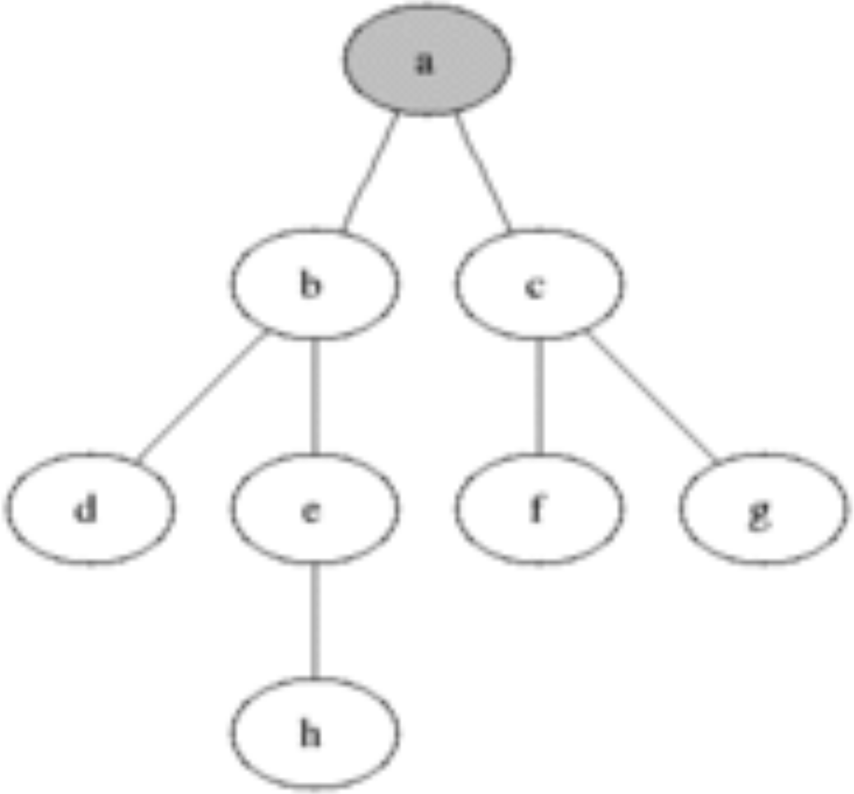

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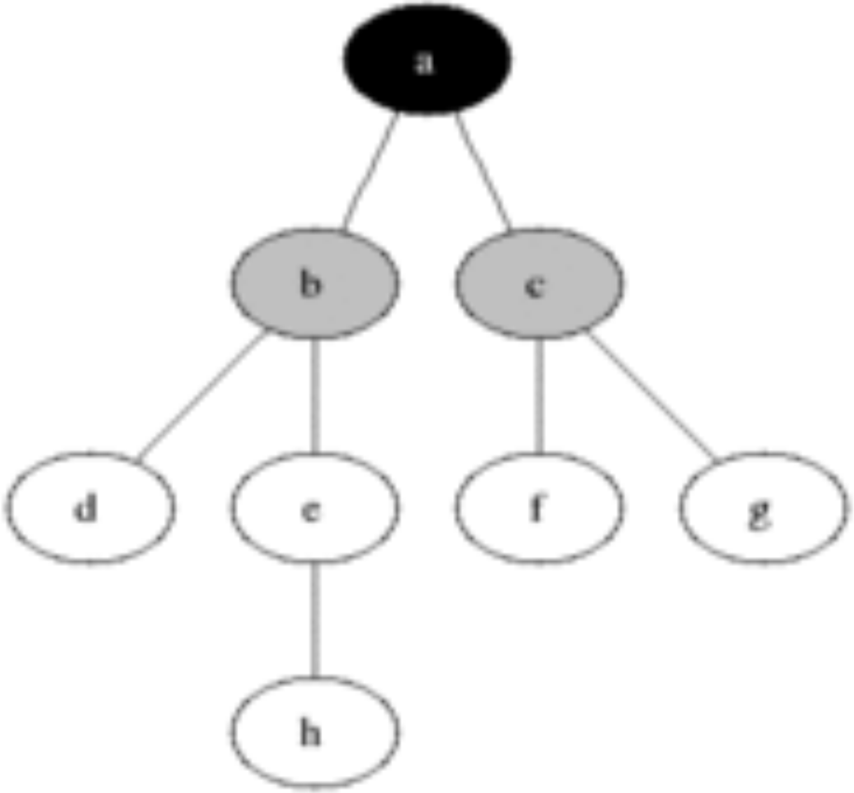

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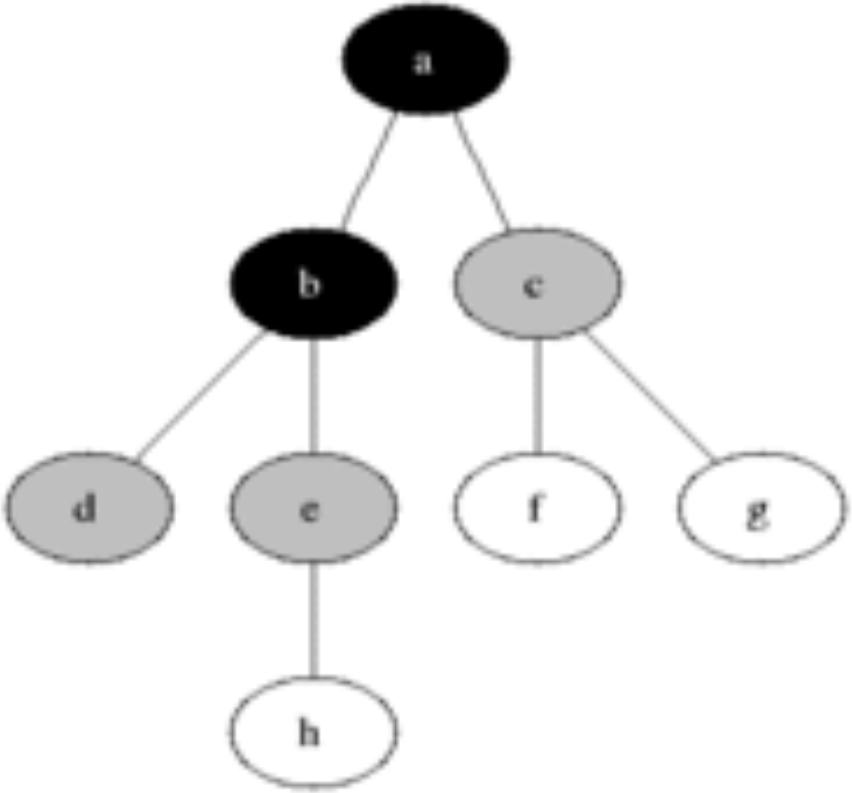

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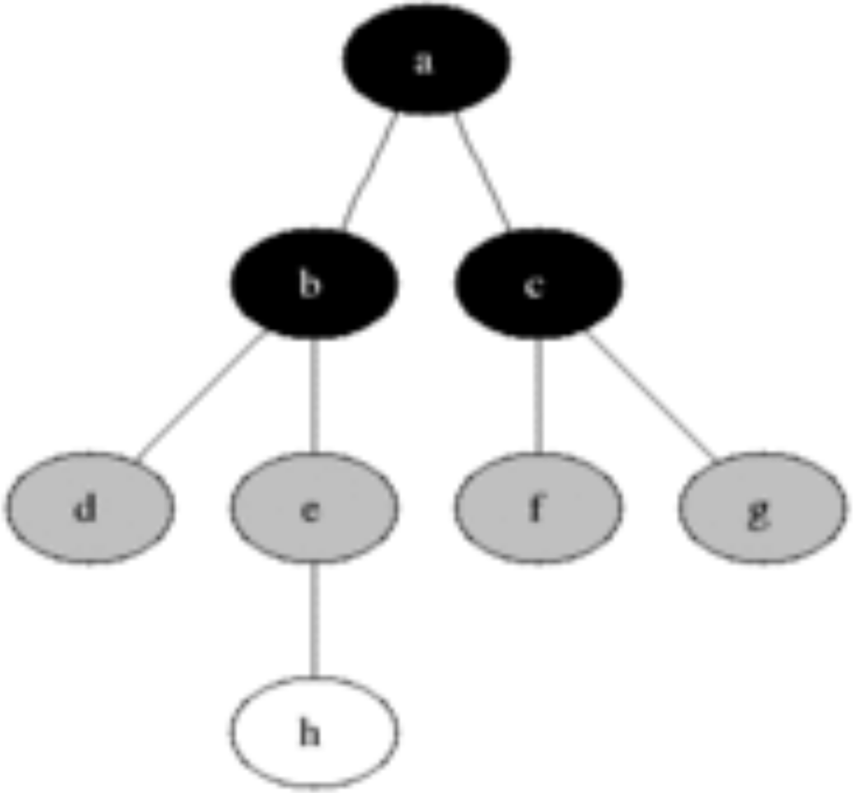

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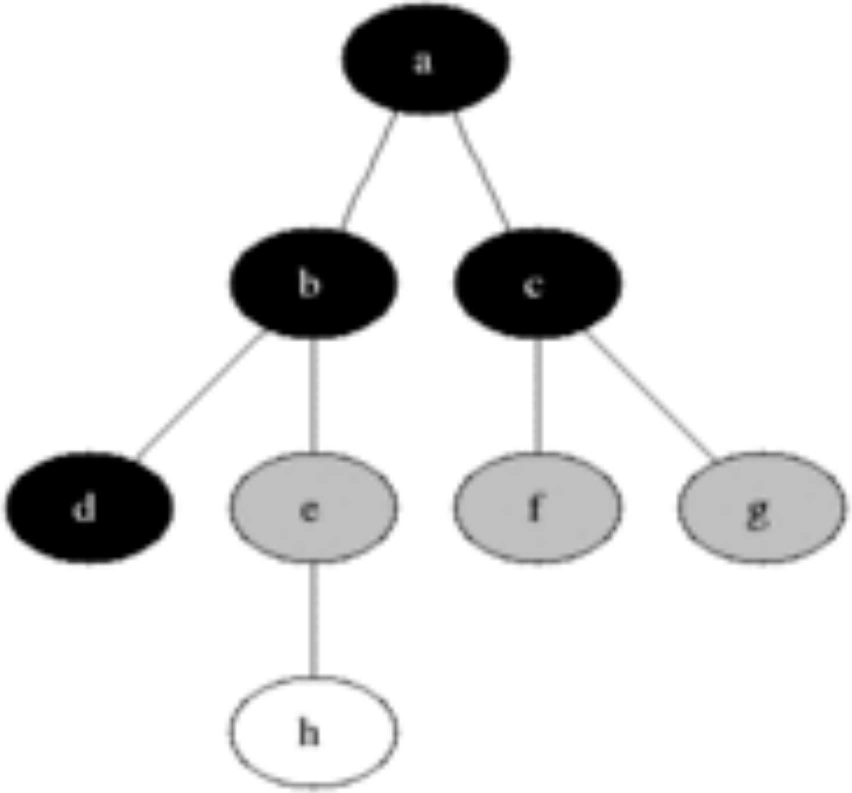

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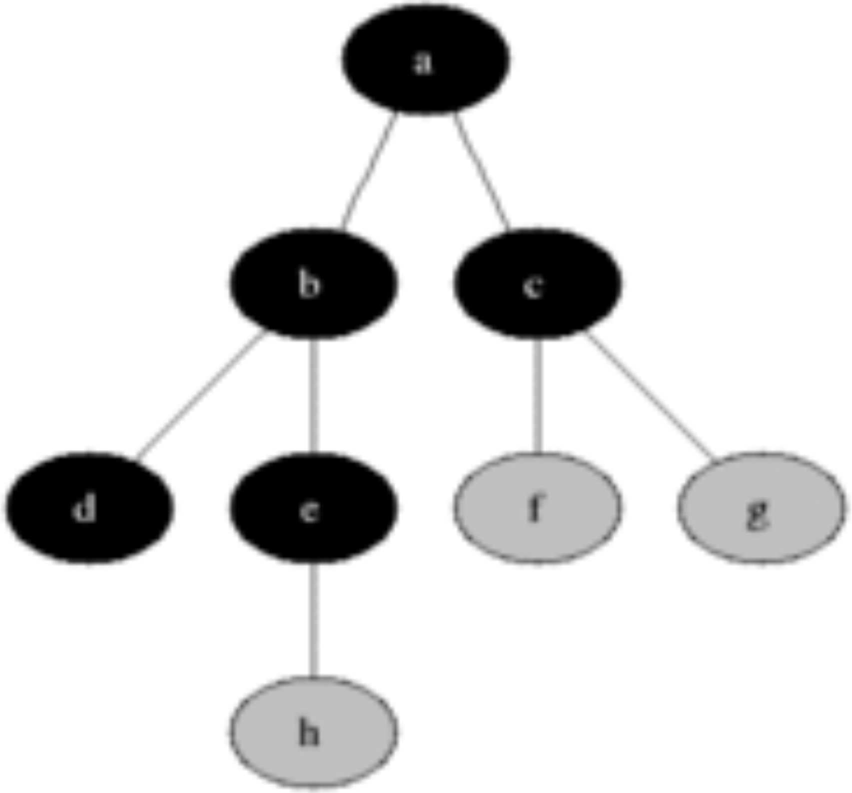

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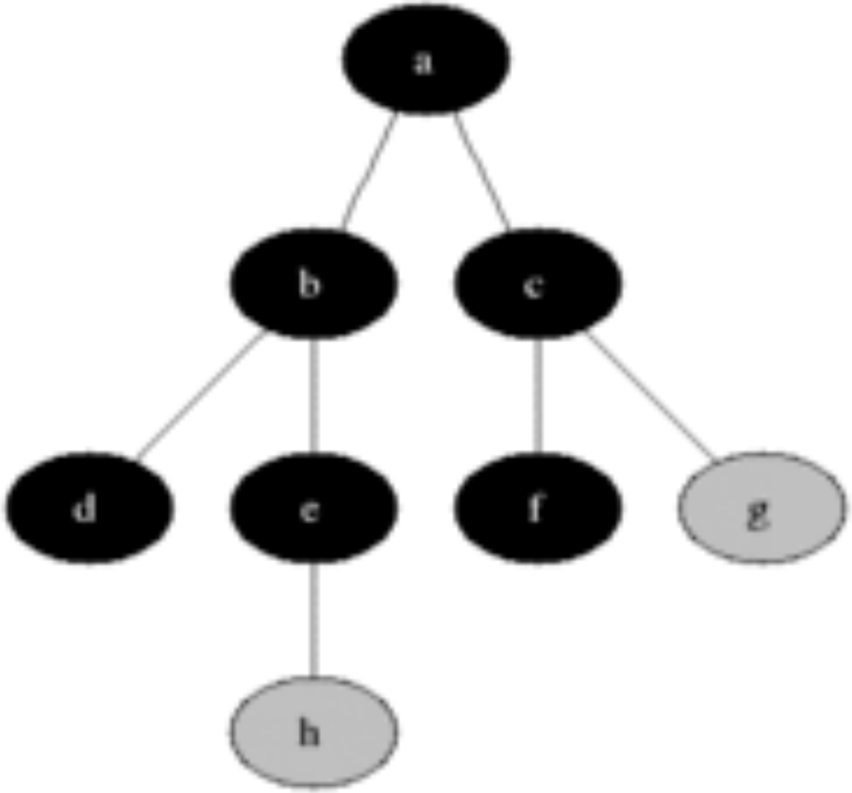

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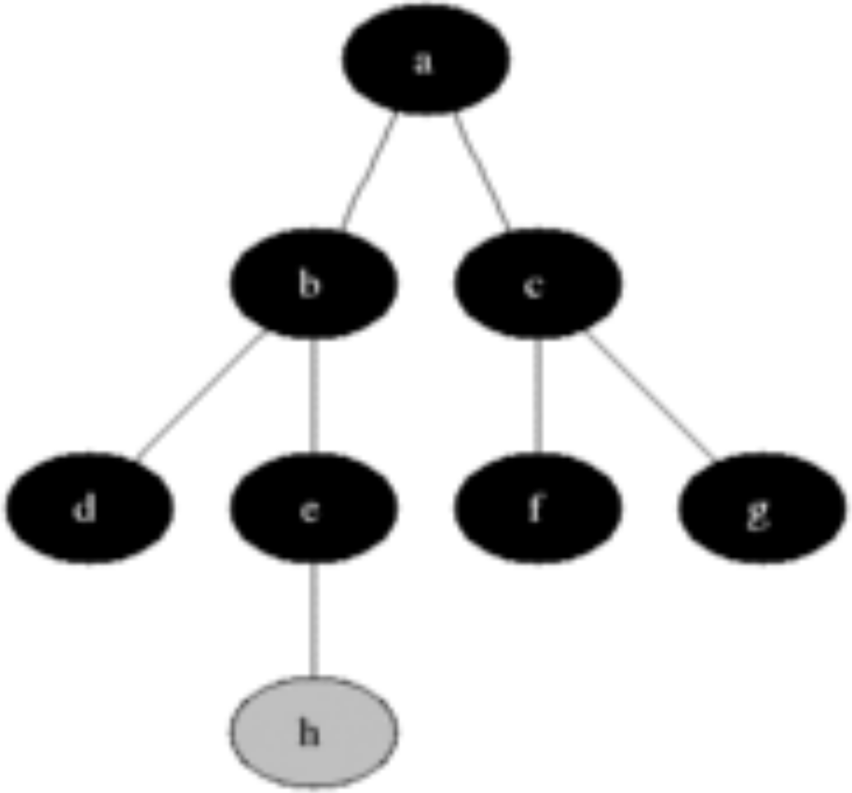

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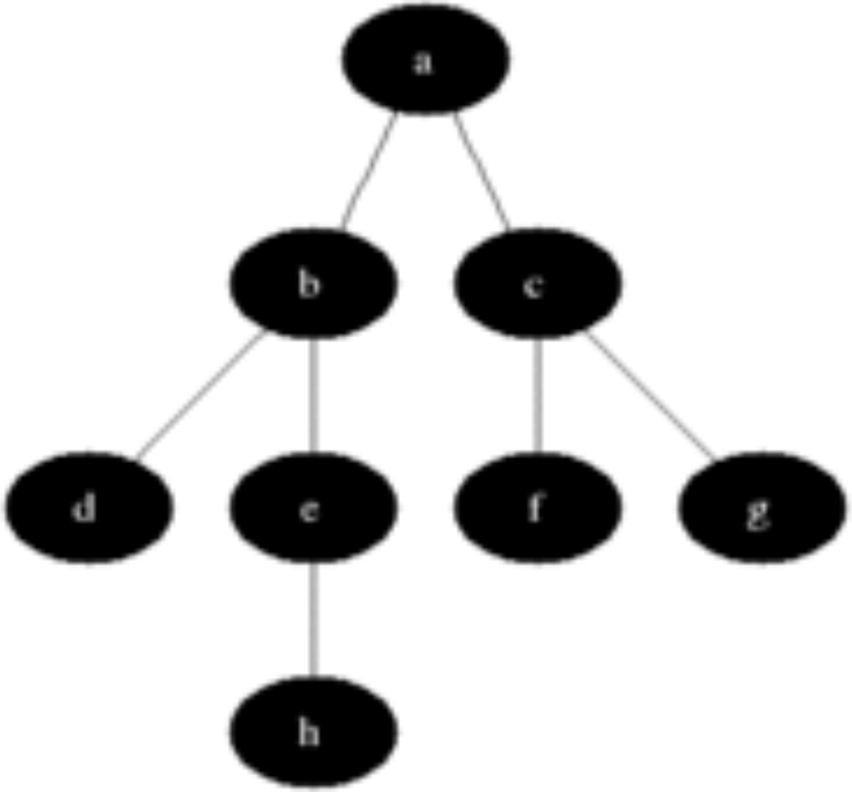

BFS Pseudocode

Idea: explore with a queue (FIFO)

```
BFS(G = (V, E), s) {
    Set mark(s) = True;
    Set mark(v) = False for all v ∈ V ∖ {s};
    Enqueue(s);
    while(queue not empty) {
        v = Dequeue();
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- $O(n)$ for initialization
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    - Examine every edge twice: when each endpoint dequeued
    - Or (equivalent): Adjacency list scanned only when vertex dequeued

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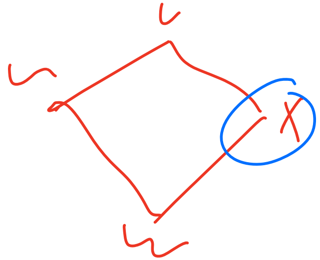

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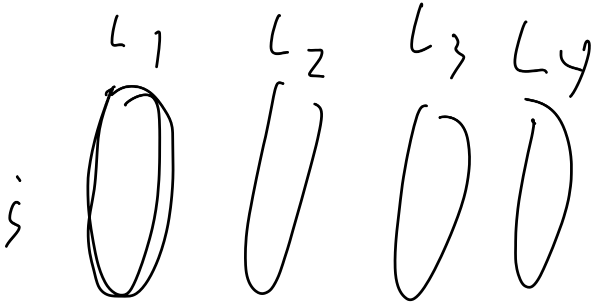
Note: edges that cause a node to be enqueued form a tree!

Correctness / Shortest Paths

Definition: Distance $\boldsymbol{d(u,v)}$ from $\boldsymbol{u}$ to $\boldsymbol{v}$ is min # edges in any path from $\boldsymbol{u}$ to $\boldsymbol{v}$

Theorem (informal): BFS($\boldsymbol{s}$) gives shortest paths from $\boldsymbol{s}$ to all other nodes



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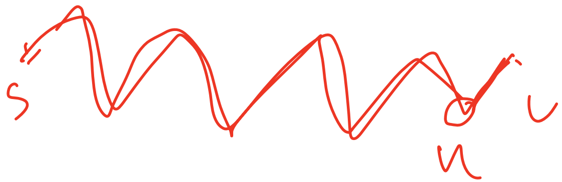

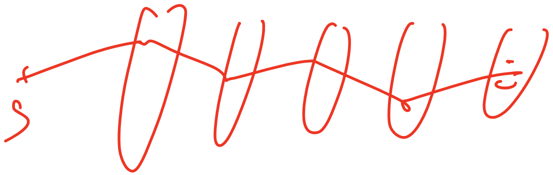

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  $\implies \{u, v\} \in T \implies v$ at layer i of T.

Depth-First Search (DFS)

DFS: Definition

Intuition: Instead of exploring wide (breadth), explore far (deep): just keep walking until see a node we've already seen, then backtrack!

Init: for each $\boldsymbol{v} \in \boldsymbol{V}$, $\boldsymbol{mark}(\boldsymbol{v}) = \boldsymbol{False}$;

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DFS(v) {
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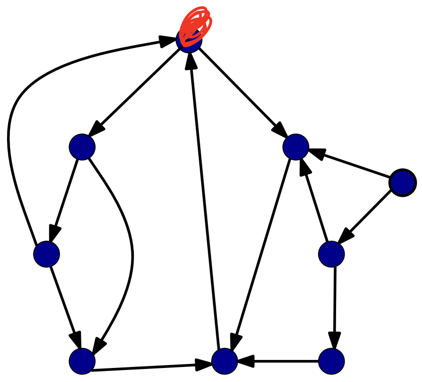

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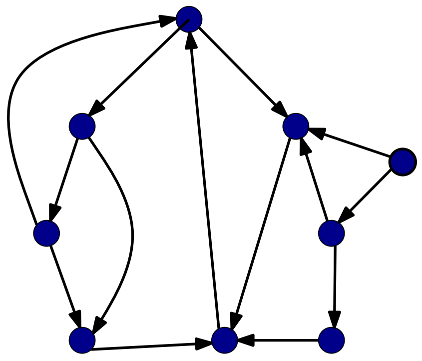

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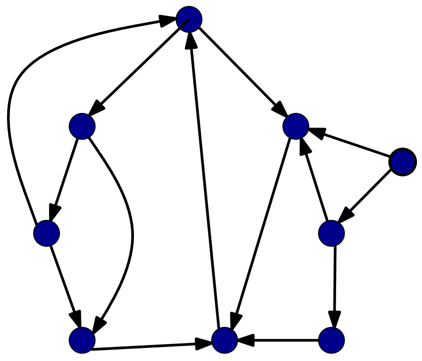

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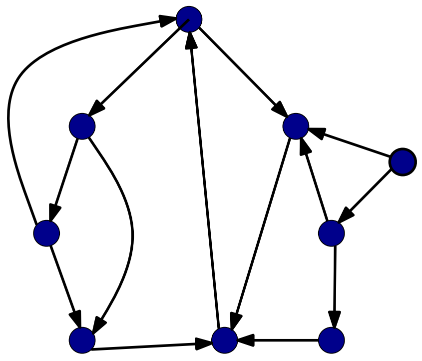

Running time: $O(m + n)$

- $O(n)$ initialization
- Every edge considered at most twice

DFS: Correctness

Definition: u is *reachable* from v if there is a path $v = v_0, v_1, \ldots, v_k = u$ such that $(v_i, v_{i+1}) \in E$ for all $i \in \{0, 1, \ldots, k-1\}$.

Theorem

When DFS(v) terminates, it has visited (marked) all nodes that are reachable from v.

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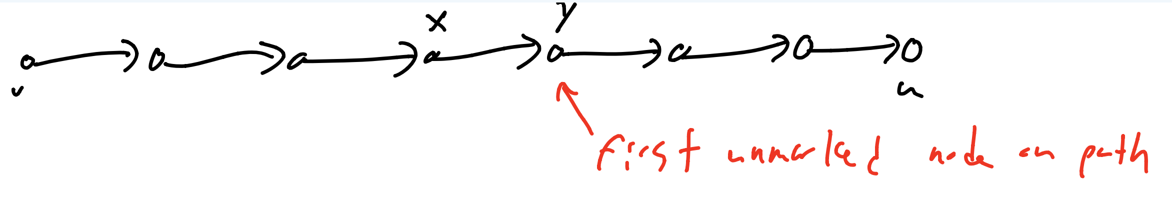

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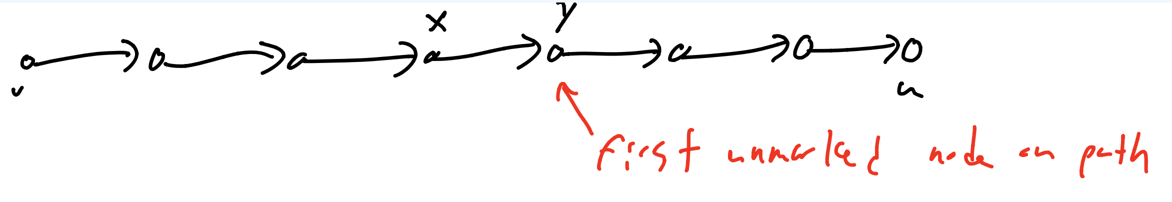


x is marked so DFS(x) must have been called

DFS: Correctness

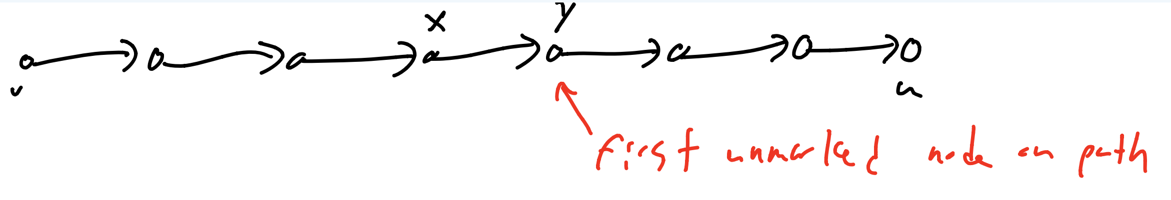
Definition: u is *reachable* from v if there is a path $v = v_0, v_1, \ldots, v_k = u$ such that $(v_i, v_{i+1}) \in E$ for all $i \in \{0, 1, \ldots, k-1\}$.

Theorem

When DFS(v) terminates, it has visited (marked) all nodes that are reachable from v.

Proof.

Suppose u reachable from v but not marked when DFS(v) terminates.



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$\implies$ y was either marked or DFS(y) called and it became marked.

DFS: Correctness

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Theorem

When DFS($\boldsymbol{v}$) terminates, it has visited (marked) all nodes that are reachable from $\boldsymbol{v}$.

Proof.

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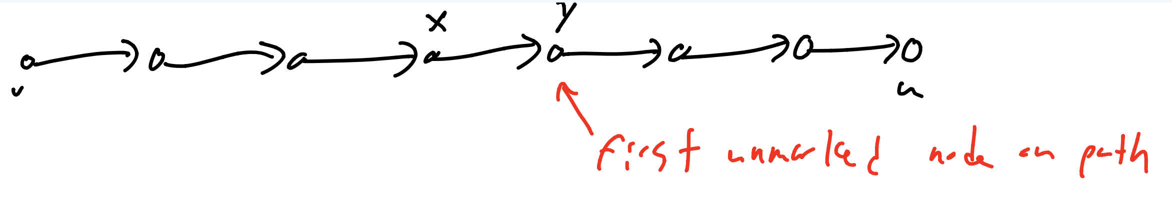


$\boldsymbol{x}$ is marked so DFS($\boldsymbol{x}$) must have been called

$\implies$ $\boldsymbol{y}$ was either marked or DFS($\boldsymbol{y}$) called and it became marked.

Contradiction. □

Graph variant

After DFS($\boldsymbol{v}$), node marked if and only if reachable from $\boldsymbol{v}$.

Might want to continue until all nodes marked.

```
DFS(G) {
    for all v ∈ V, set mark(v) = False;
    while there exists an unmarked node v {
        DFS(v);
    }
}
```

Timestamps

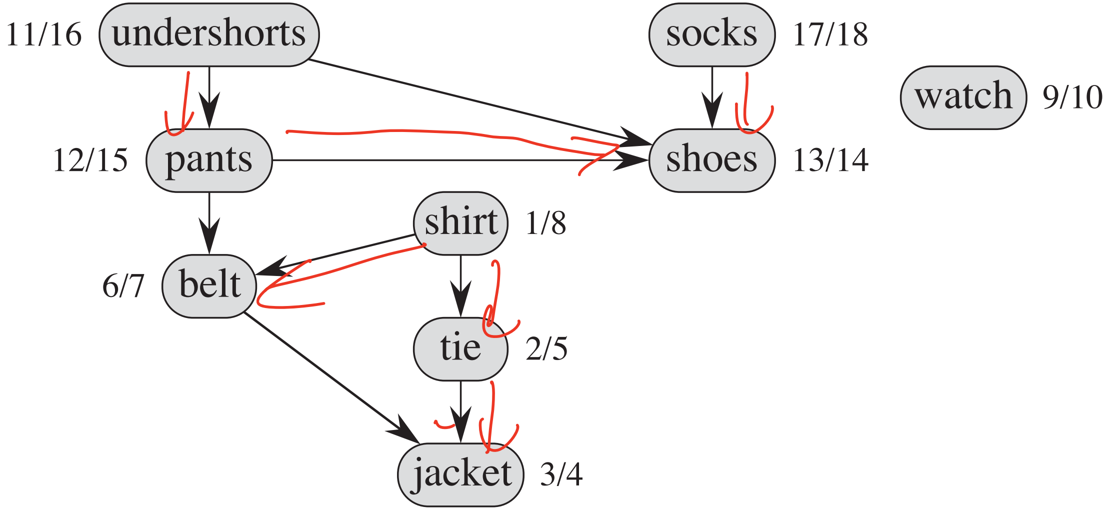
Explicitly keep track of "start" and "finishing" times

- Replaces ***mark***

```
DFS(G) {
    t = 0;
    for all v ∈ V {
        start(v) = 0;
        finish(v) = 0;
    }
    while ∃v ∈ V with start(v) = 0 {
        DFS(v);
    }
}
```

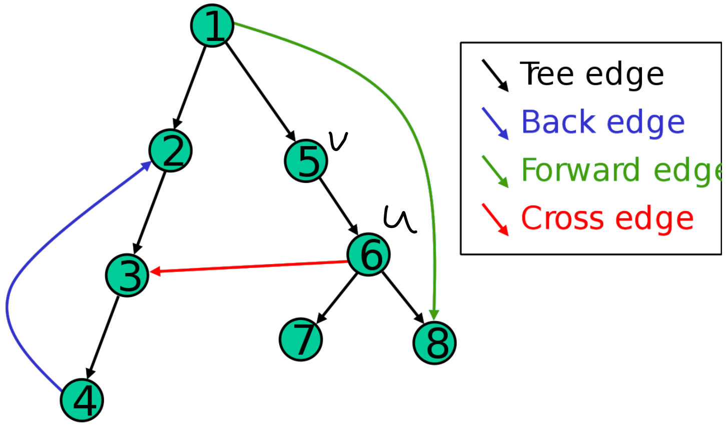
```
DFS(v) {
    t = t + 1;
    start(v) = t;
    for each edge (v, u) ∈ A[v] {
        if start(u) == 0 then DFS(u);
    }
    t = t + 1;
    finish(v) = t;
}
```

Timestamp Example



Edge Types

DFS naturally gives a spanning forest: edge (v, u) if DFS(v) calls DFS(u)



Forward Edges: (v, u) such that u descendent of v (includes tree edges)

Back Edges: (v, u) such that u an ancestor of v

Cross Edges: (v, u) such that u neither a descendent nor an ancestor of v

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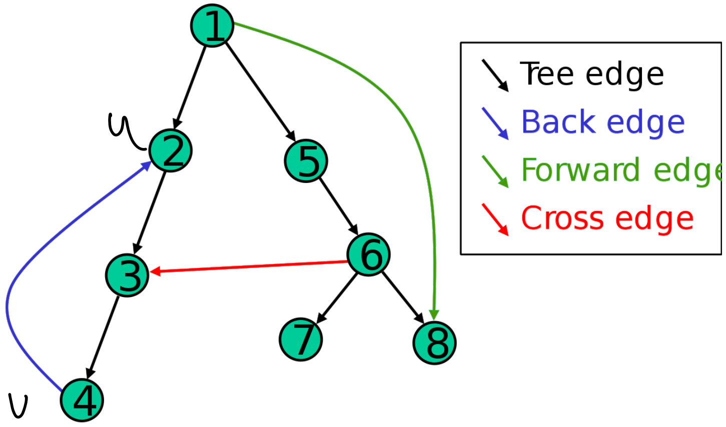

Forward Edges: $(\boldsymbol{v}, \boldsymbol{u})$ such that $\boldsymbol{u}$ descendent of $\boldsymbol{v}$ (includes tree edges)

$\boldsymbol{start}(\boldsymbol{v}) < \boldsymbol{start}(\boldsymbol{u}) < \boldsymbol{finish}(\boldsymbol{u}) < \boldsymbol{finish}(\boldsymbol{v})$

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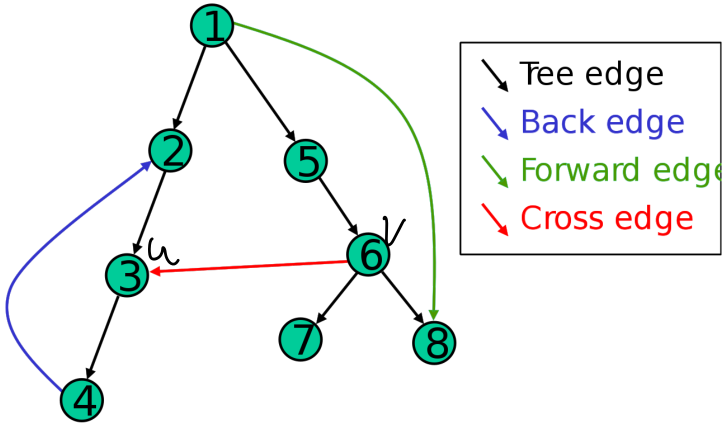

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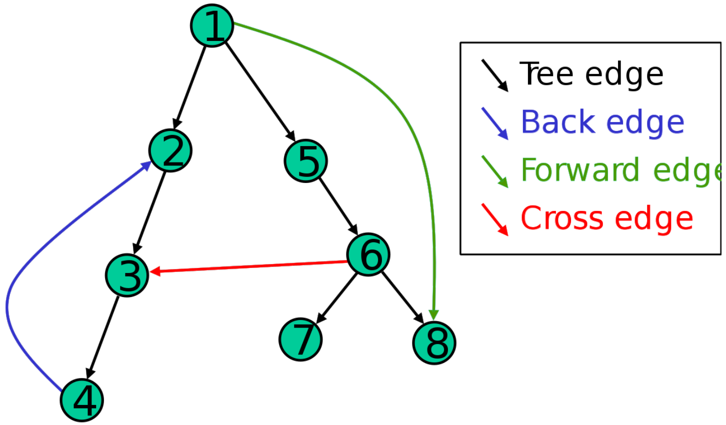

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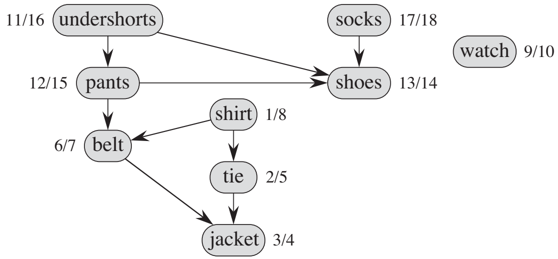
Topological Sort

Definitions

Definition

A directed graph G is a *Directed Acyclic Graph (DAG)* if it has no directed cycles.



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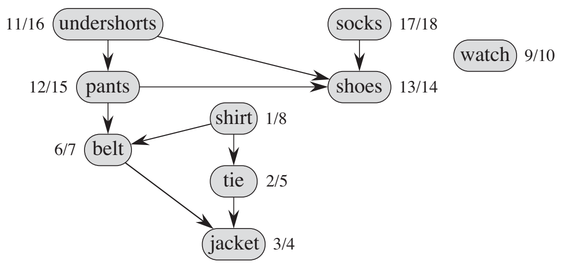
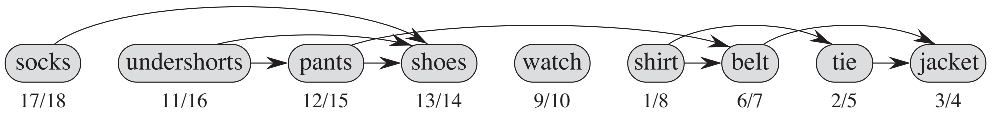


Definition

A *topological sort* $v_1, v_2, \ldots, v_n$ of a DAG is an ordering of the vertices such that all edges are of the form (v_i, v_j) with $i < j$.



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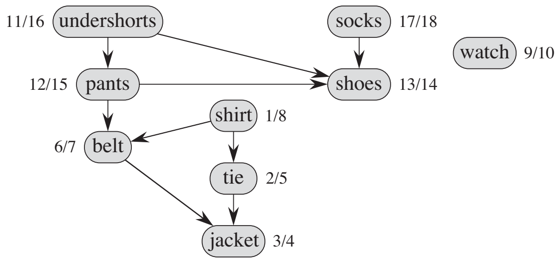

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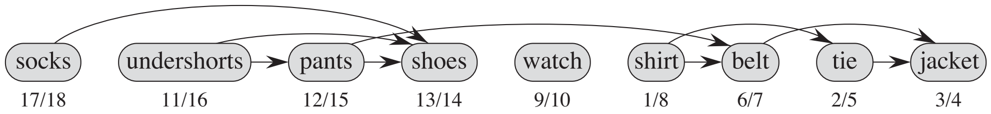

Q: Can we always topological sort a DAG? How fast?

Topological Sort

Algorithm (informal): Run DFS($\boldsymbol{G}$). When DFS($\boldsymbol{v}$) returns, put $\boldsymbol{v}$ at beginning of list

Topological Sort

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```
DFS(G) {
    list → head = NULL;
    t = 0;
    for all v ∈ V {
        start(v) = 0;
        finish(v) = 0;
    }
    while ∃v ∈ V with start(v) = 0 {
        DFS(v);
    }
}
```

```
DFS(v) {
    t = t + 1;
    start(v) = t;
    for each edge (v, u) ∈ A[v] {
        if start(u) == 0 then DFS(u);
    }
    t = t + 1;
    finish(v) = t;
    temp = list → head;
    list → head = v;
    list → head → next = temp;
}
```

Characterizing DAGs

Theorem

A directed graph G is a DAG if and only if DFS(G) has no back edges.

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Only if ($\Rightarrow$): contrapositive. If $\boldsymbol{G}$ has a back edge:

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If ($\Leftarrow$): contrapositive. If $\boldsymbol{G}$ has a directed cycle $\boldsymbol{C}$:

Characterizing DAGs

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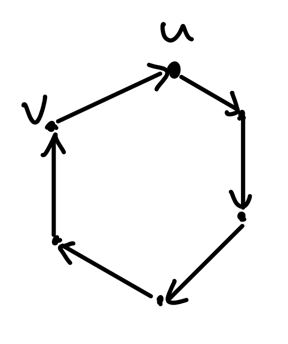
A directed graph G is a DAG if and only if DFS(G) has no back edges.

Proof.

Only if ($\Rightarrow$): contrapositive. If G has a back edge: Directed cycle! Not a DAG.

If ($\Leftarrow$): contrapositive. If G has a directed cycle C:

- Let $u \in C$ with minimum start value, v predecessor in cycle
- All nodes in C reachable from $u \implies$ all nodes in C descendants of u
- (v, u) a back edge



$\square$

Topological Sort Analysis

Correctness: Since G a DAG, never see back edge

$\implies$ Every edge (v, u) out of v a forward or cross edge

$\implies$ $\textit{finish}(u) < \textit{finish}(v)$

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Running Time: Same as DFS! $O(m + n)$