Lecture 15: Basic Graph Algorithms II

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October 14, 2025

601.433/633 Introduction to Algorithms

Introduction

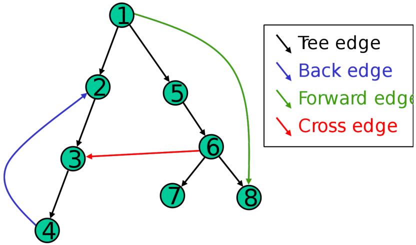
Last time: BFS and DFS

Today: Topological Sort, Strongly Connected Components

- Both very classical and important uses of DFS!

Edge Types

DFS naturally gives a spanning forest: edge (v, u) if DFS(v) calls DFS(u)



Forward Edges: (v, u) such that u descendent of v (includes tree edges)

$start(v) < start(u) < finish(u) < finish(v)$

Back Edges: (v, u) such that u an ancestor of v

$start(u) < start(v) < finish(v) < finish(u)$

Cross Edges: (v, u) such that u neither a descendent nor an ancestor of v

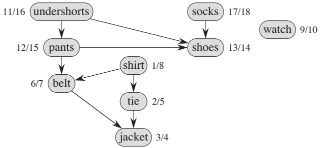
$start(u) < finish(u) < start(v) < finish(v)$

Topological Sort

Definitions

Definition

A directed graph G is a *Directed Acyclic Graph (DAG)* if it has no directed cycles.



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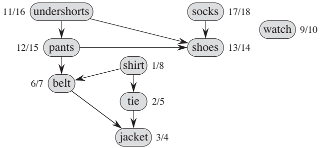

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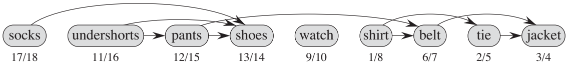
A *topological sort* $v_1, v_2, \ldots, v_n$ of a DAG is an ordering of the vertices such that all edges are of the form (v_i, v_j) with $i < j$.



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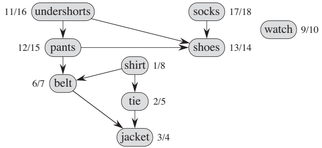

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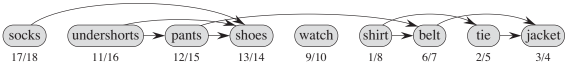

Q: Can we always topological sort a DAG? How fast?

Topological Sort

Algorithm (informal): Run DFS($\boldsymbol{G}$). When DFS($\boldsymbol{v}$) returns, put $\boldsymbol{v}$ at beginning of list

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```
DFS(G) {
    list → head = NULL;
    t = 0;
    for all v ∈ V {
        start(v) = 0;
        finish(v) = 0;
    }
    while ∃v ∈ V with start(v) = 0 {
        DFS(v);
    }
}
```

```
DFS(v) {
    t = t + 1;
    start(v) = t;
    for each edge (v, u) ∈ A[v] {
        if start(u) == 0 then DFS(u);
    }
    t = t + 1;
    finish(v) = t;
    temp = list → head;
    list → head = v;
    list → head → next = temp;
}
```

Characterizing DAGs

Theorem

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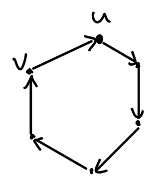
If ($\Leftarrow$): contrapositive. If $\boldsymbol{G}$ has a directed cycle $\boldsymbol{C}$:

- Let $\boldsymbol{u} \in \boldsymbol{C}$ with minimum start value, $\boldsymbol{v}$ predecessor in cycle
- All nodes in $\boldsymbol{C}$ reachable from $\boldsymbol{u} \implies$ all nodes in $\boldsymbol{C}$ descendants of $\boldsymbol{u}$
- $(\boldsymbol{v}, \boldsymbol{u})$ a back edge



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Topological Sort Analysis

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Running Time:

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Running Time: Same as DFS! $O(m + n)$

Strongly Connected Components (SCC)

Definitions

Another application of DFS. "Kosaraju's Algorithm": Developed by Rao Kosaraju, professor emeritus at JHU CS!

$G = (V, E)$ a directed graph.

Definition

$C \subseteq V$ is a *strongly connected component (SCC)* if it is a *maximal* subset such that for all $u, v \in C$, u can reach v and vice versa (bireachable).

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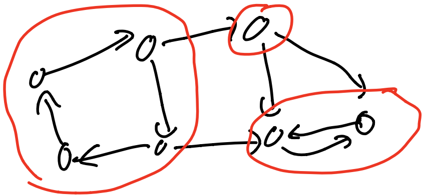

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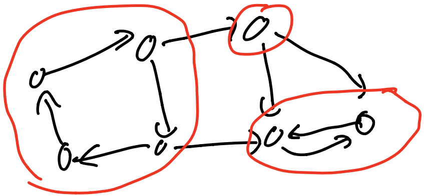

Fact: There is a *unique* partition of V into SCCs

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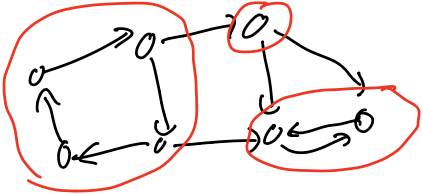

Fact: There is a *unique* partition of V into SCCs

Proof: Bireachability is an equivalence relation: if u and v are bireachable, and v and w are bireachable, then u and w are bireachable.

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Problem: Given G, compute SCCs (partition V into the SCCs)

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- Running time: $O(n(m+n))$

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Can we do better? $O(m+n)$?

Graph of SCCs

Definition: Let $\hat{G}$ be graph of SCCs:

- Vertex $v(C)$ for each SCC C
- Edge $(v(C), v(C'))$ if $\exists\ u \in C, v \in C'$ such that $(u, v) \in E$

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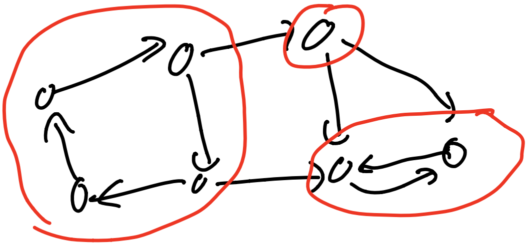

Graph of SCCs: Structure

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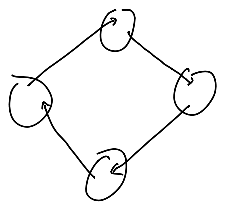

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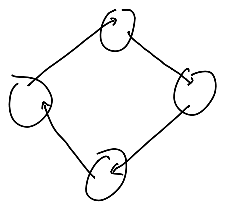

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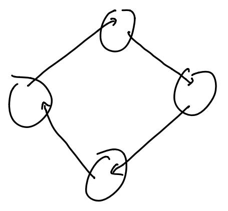

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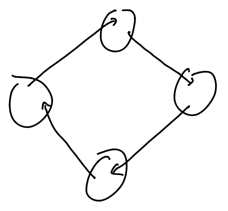
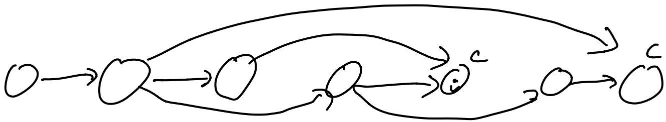
Contradiction!



Sink SCC

Since $\hat{\boldsymbol{G}}$ a DAG, has a topological sort



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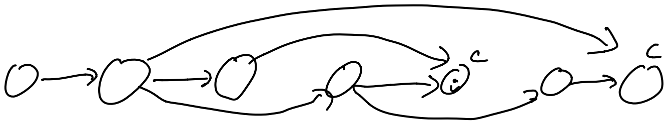

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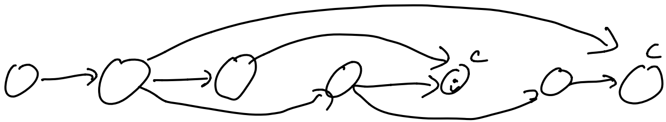

Definition: SCC C is a *sink* SCC if no outgoing edges, *source* SCC if no incoming edges

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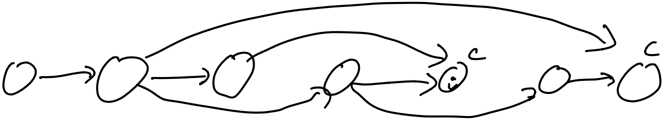

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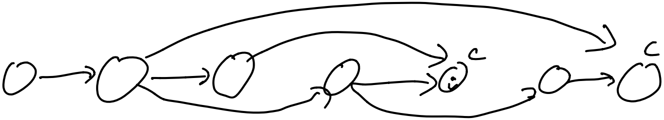

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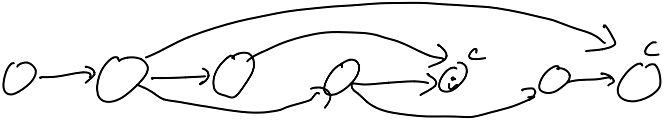

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What happens if we run DFS(v) where v in a sink SCC?

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Strategy: find node in sink SCC, run DFS, remove nodes found, repeat

SCCs and DFS

Run DFS(G), and let $\textit{finish}(C) = \max_{v \in C} \textit{finish}(v)$

Lemma

Let C_1, C_2 distinct SCCs s.t. $(v(C_1), v(C_2)) \in E(\hat{G})$. Then $\textit{finish}(C_1) > \textit{finish}(C_2)$.

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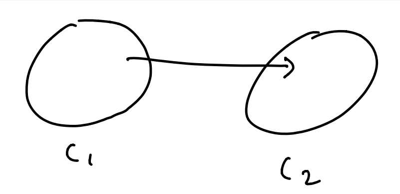

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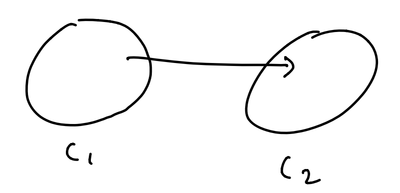

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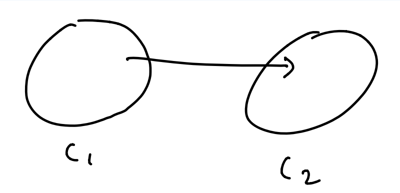

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Let C_1, C_2 *distinct SCCs s.t.* $(v(C_1), v(C_2)) \in E(\hat{G})$. *Then* $\boldsymbol{finish}(C_1) > \boldsymbol{finish}(C_2)$.

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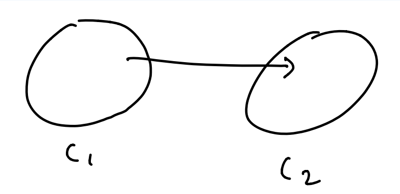

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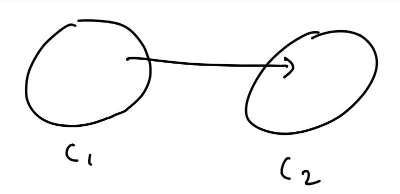

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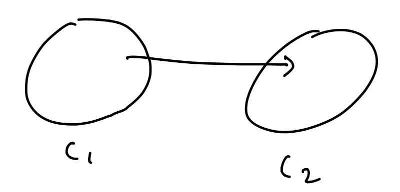

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So node of *max* finishing time in a *source* SCC (no incoming edges in $\hat{G}$).

Useful Corollary

Run DFS(G) to get finish times.

Corollary

Let $\mathcal{C}$ be collection of all SCCs of G, and let $\mathcal{C}' \subseteq \mathcal{C}$. Let $G' = G \setminus \left(\bigcup_{C \in \mathcal{C}'} C\right)$. Then the node $v = \mathbf{argmax}_{u \in \bigcup_{C \in \mathcal{C} \setminus \mathcal{C}'} C} \mathit{finish}(u)$ is in an SCC of G that is a source SCC of G'.

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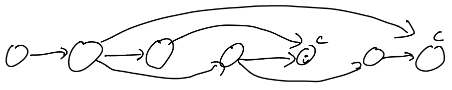
Proof Sketch.

Clearly SCCs of $\boldsymbol{G}'$ are precisely $\mathcal{C} \setminus \mathcal{C}'$:



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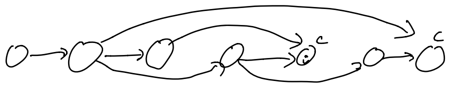

Suppose v in SCC C not source SCC of G'

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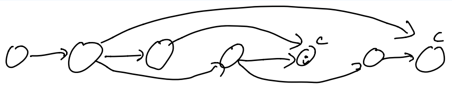

Suppose v in SCC C not source SCC of G'

$\implies$ exists edge (u', v') in G' where $v' \in C$ but u' in different SCC C'

Useful Corollary

Run DFS(G) to get finish times.

Corollary

Let $\mathcal{C}$ be collection of all SCCs of G, and let $\mathcal{C}' \subseteq \mathcal{C}$. Let $G' = G \setminus (\bigcup_{C \in \mathcal{C}'} C)$. Then the node $v = \mathbf{argmax}_{u \in \bigcup_{C \in \mathcal{C} \setminus \mathcal{C}'} C} \mathbf{finish}(u)$ is in an SCC of G that is a source SCC of G'.

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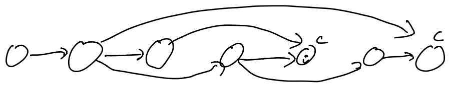

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Lemma $\Longrightarrow$ $\mathbf{finish}(C') > \mathbf{finish}(C)$, contradiction to def of v. □

Kosaraju's Algorithm

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- DFS(G^T) to get finishing times and order π on V from largest finishing time to smallest
- Set $mark(v) = False$ for all $v \in V$
- Forall $v \in V$ in order of π {
    - if $mark(v) = False$ {
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Running Time:

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Running Time: $O(m + n)$

Correctness Sketch

Let $C_1, C_2, \ldots, C_k$ be sets identified by algorithm (in order)

Theorem

C_i is a sink SCC of $G \setminus \left(\bigcup_{j=1}^{i-1} C_j\right)$, and an SCC of G.

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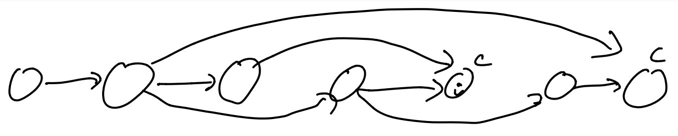

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Inductive case: Let $i > 1$. Let v unmarked node with largest finishing time.

- By induction, subgraph of unmarked nodes is G minus $i - 1$ SCCs of G
- Corollary $\implies v$ must be in sink SCC of unmarked nodes so get an SCC of unmarked nodes when run DFS
- Corollary $\implies$ SCC of original graph