Lecture 18: Minimum Spanning Trees

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October 30, 2025
601.433/633 Introduction to Algorithms

Introduction

Definition

A *spanning tree* of an undirected graph $G = (V, E)$ is a set of edges $T \subseteq E$ such that (V, T) is connected and acyclic.

Definition

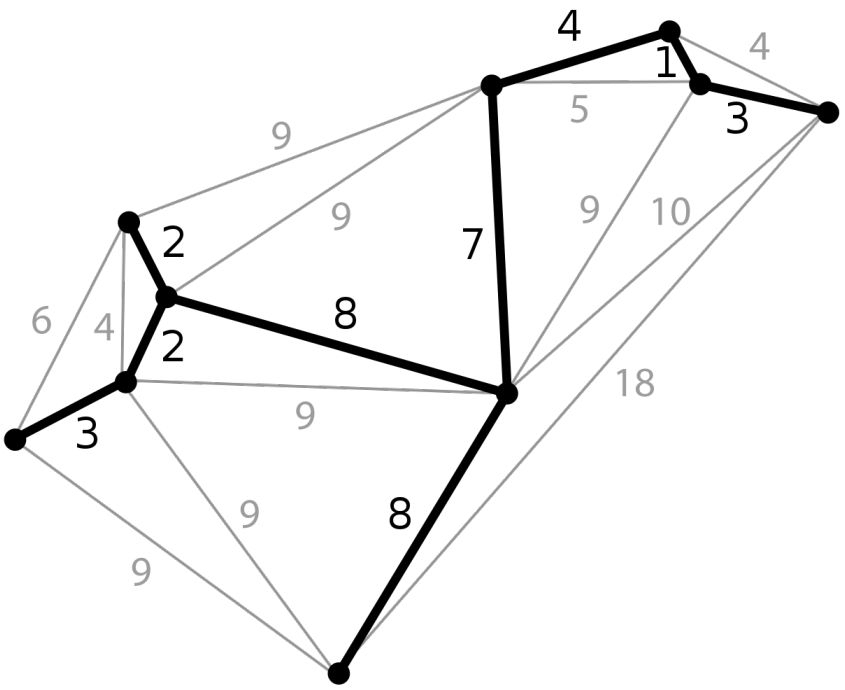
Minimum Spanning Tree problem (MST):

- Input:
  - Undirected graph $G = (V, E)$
  - Edge weights $w : E \to \mathbb{R}_{\geq 0}$
- Output: Spanning tree minimizing $w(T) = \sum_{e \in T} w(e)$.

Foundational problem in *network design*. Tons of applications.

Today: one "recipe", two different algorithms from recipe. Main idea: greedy.

Examples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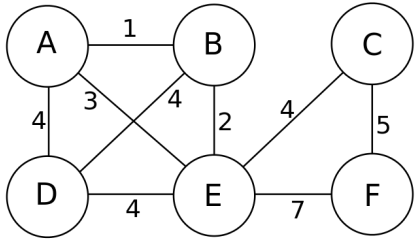
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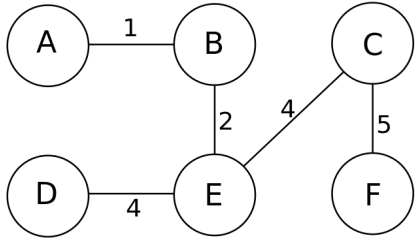
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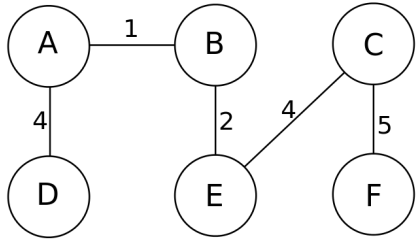
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Generic Algorithm

Generic Greedy

Definition

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Claim: A always a subset of some MST.
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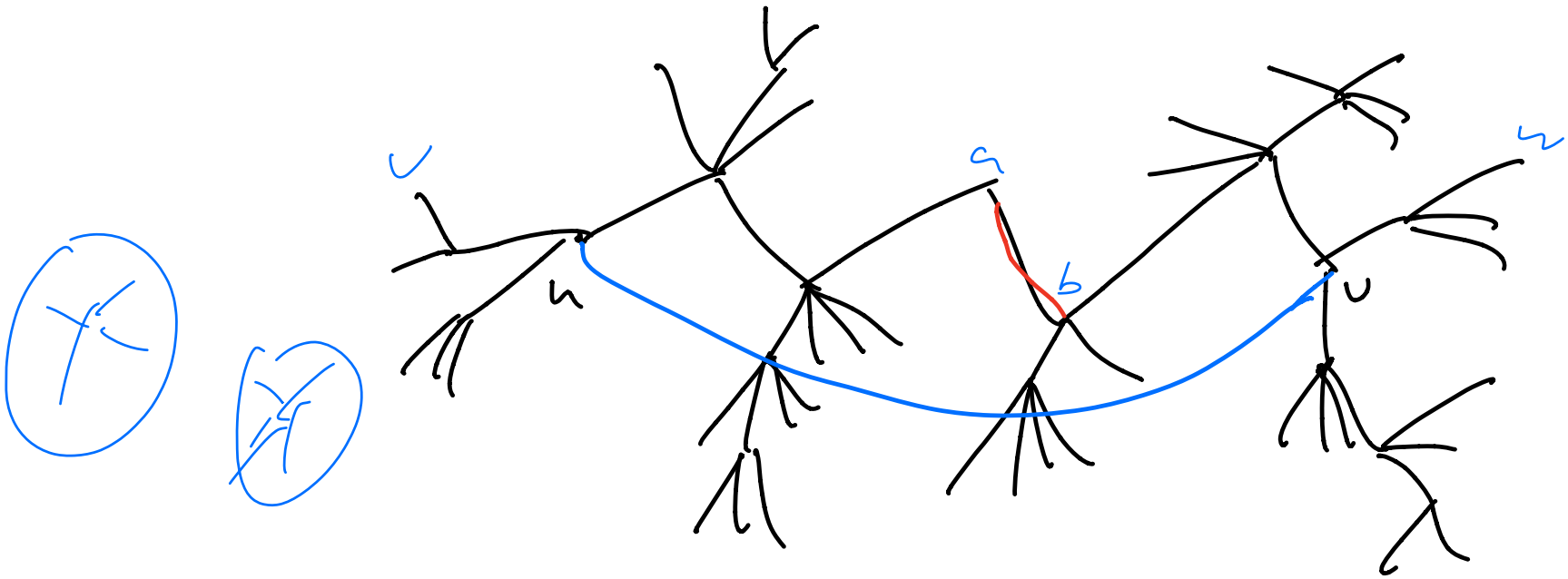
But how to find a safe edge? And which one to add?

Structural Properties

Lemma

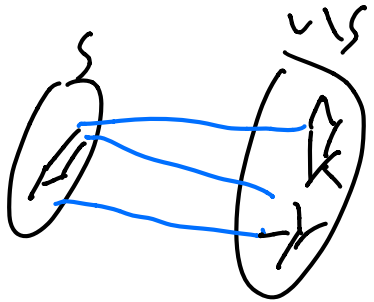
Let T *be a spanning tree, let* $u, v \in V$*, and let* P *be the* $u - v$ *path in* T*. If* $\{u, v\} \notin T$*, then* $T' = (T \cup \{\{u, v\}\}) \setminus \{e\}$ *is a spanning tree for all* $e \in P$*.*



Structural Properties

Definition

A *cut* $(S, V \setminus S)$ (or $(S, \bar{S})$ or just S) is a partition of V into two parts. Edge e *crosses* cut $(S, \bar{S})$ if e has one endpoint in S and one endpoint in $\bar{S}$.



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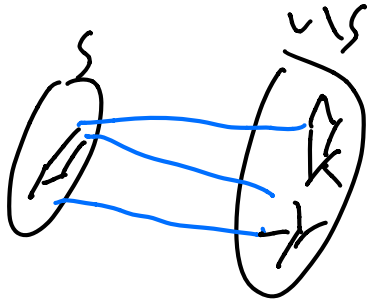

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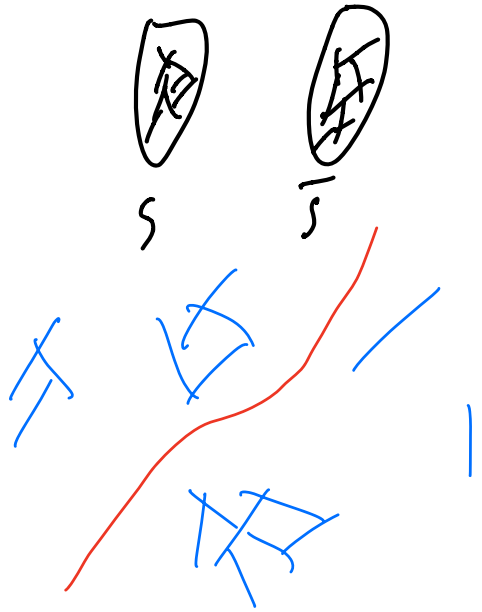
Cut $(S, \bar{S})$ *respects* $A \subseteq E$ if no edge in A crosses $(S, \bar{S})$



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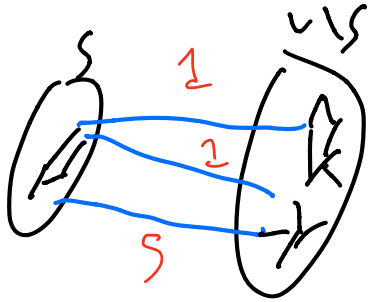

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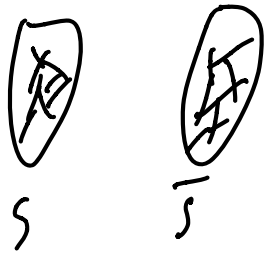


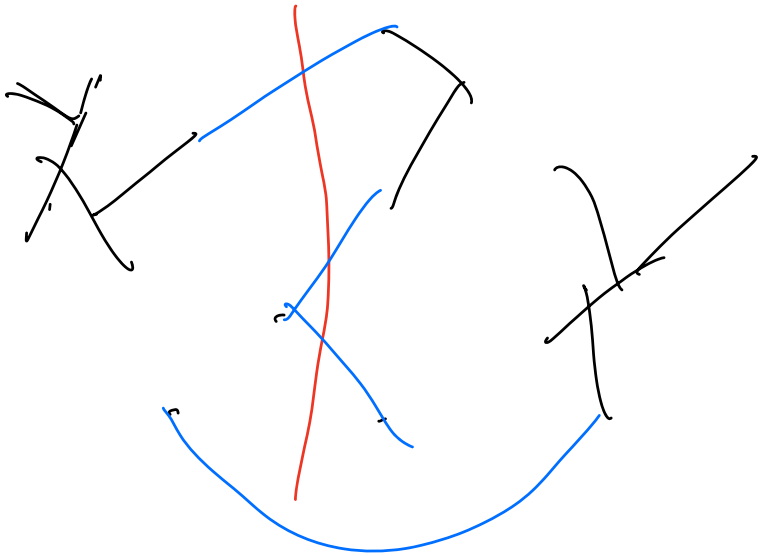
Definition

e is a *light edge* for $(S, \bar{S})$ if e crosses $(S, \bar{S})$ and $w(e) = \min_{e' \text{ crossing } (S,\bar{S})} w(e')$

Main Structural Theorem

Theorem

Let $A \subseteq E$ be a subset of some MST T, let $(S, \bar{S})$ be a cut respecting A, and let $e = \{u, v\}$ be a light edge for $(S, \bar{S})$. Then e is safe for A.



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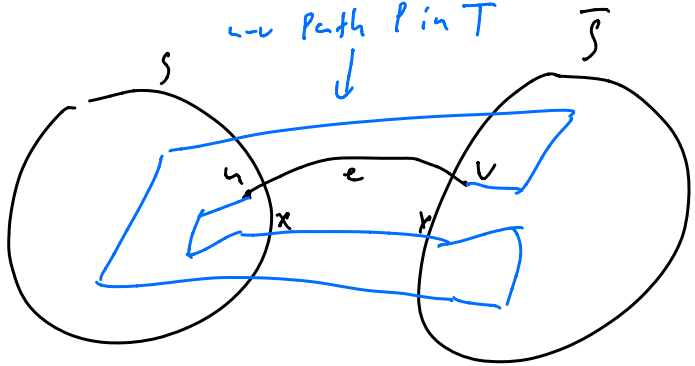
Let $T' = (T \cup \{e\}) \setminus \{\{x, y\}\}$

$\Longrightarrow$ T' a spanning tree by first lemma



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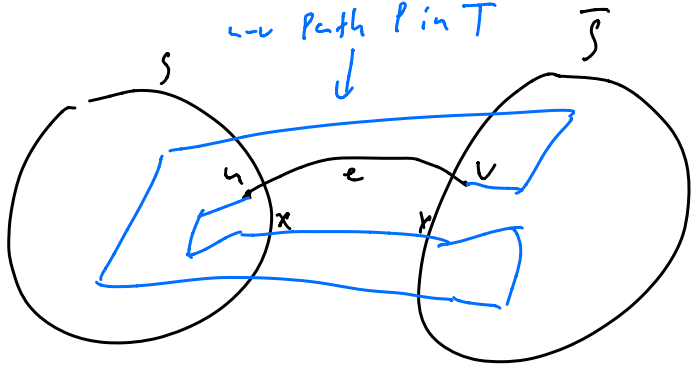

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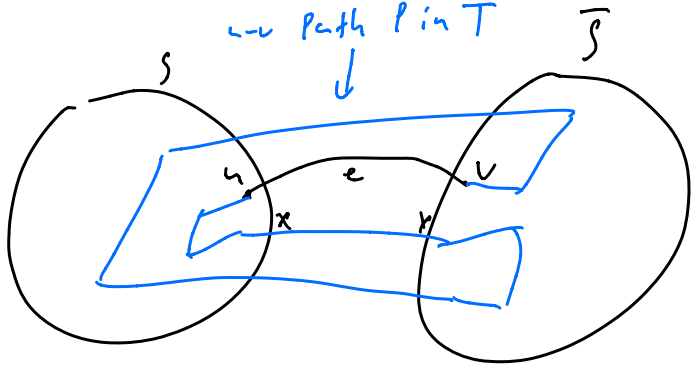

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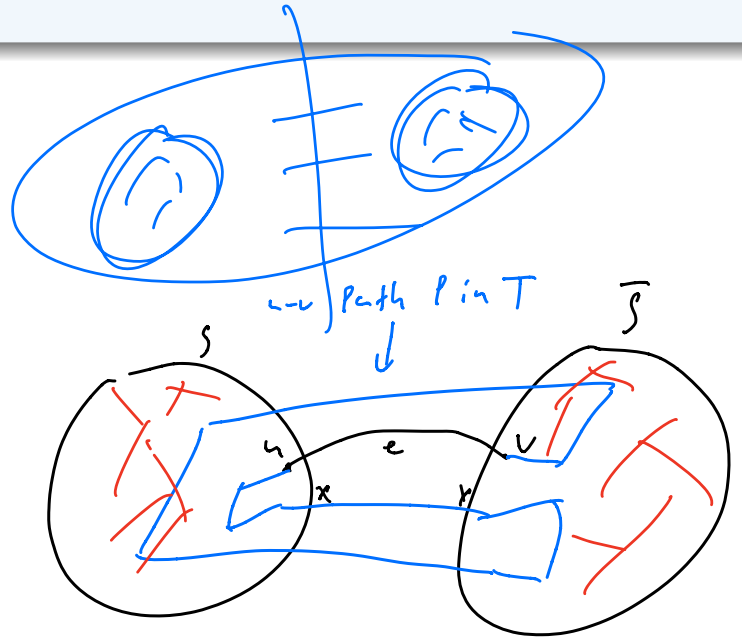

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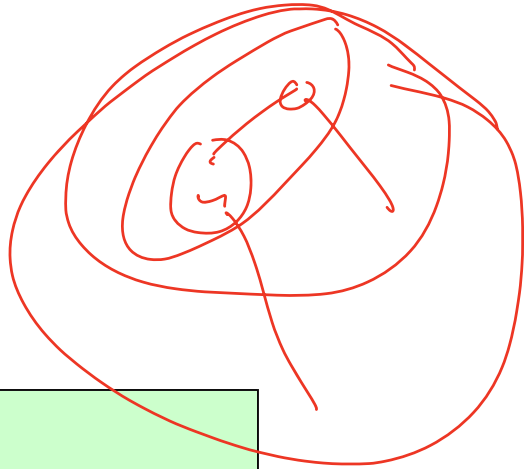

Idea: start at arbitrary node u. Greedily grow MST out of u.

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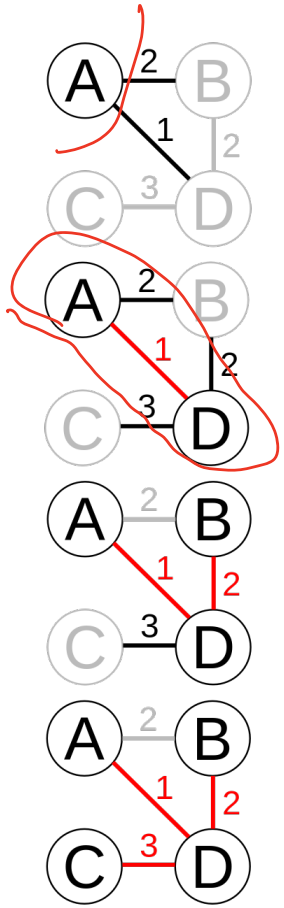

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Prim's algorithm returns an MST.

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Proof.

Just Generic-MST!

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Just Generic-MST!

- $(S, \bar{S})$ always respects A (induction).
- If edge e added then light for $(S, \bar{S})$
- Hence e safe for A by main structural theorem.

□

Running Time

Trivial analysis:

- Every spanning tree has $n-1$ edges $\implies$ $n-1$ iterations
- In each iteration, look through all edges to find min-weight edge crossing $(S, \bar{S})$ $\implies$ $O(m)$ time
- Total $O(mn)$

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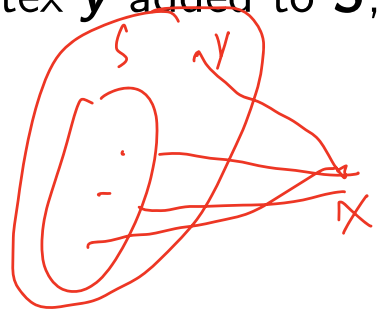

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- n Inserts, n Extract-Mins, m Decrease-Keys
- Like Dijkstra, $O(m \log n)$ using binary heap. $O(m + n \log n)$ with Fibonacci heap (only Extract-Min is logarithmic)

Kruskal's Algorithm

Algorithm

Intuition: can we be *even greedier* than Prim's Algorithm?

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$\boldsymbol{A} = \varnothing$
Sort edges by weight (small to large)
For each edge $\boldsymbol{e}$ in this order {
    if $\boldsymbol{A} \cup \{\boldsymbol{e}\}$ has no cycles, $\boldsymbol{A} = \boldsymbol{A} \cup \{\boldsymbol{e}\}$
}
return $\boldsymbol{A}$

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Theorem

Kruskal's algorithm computes an MST.

Want to show just Generic-MST: when $\{\boldsymbol{u}, \boldsymbol{v}\}$ added, it was safe for $\boldsymbol{A}$.

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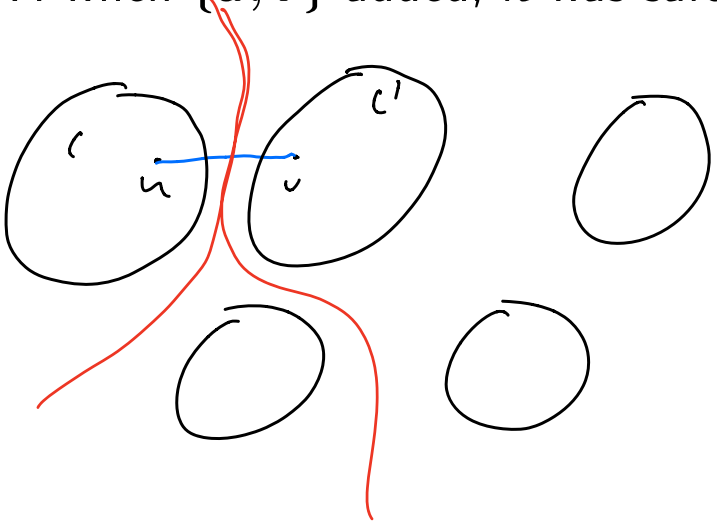

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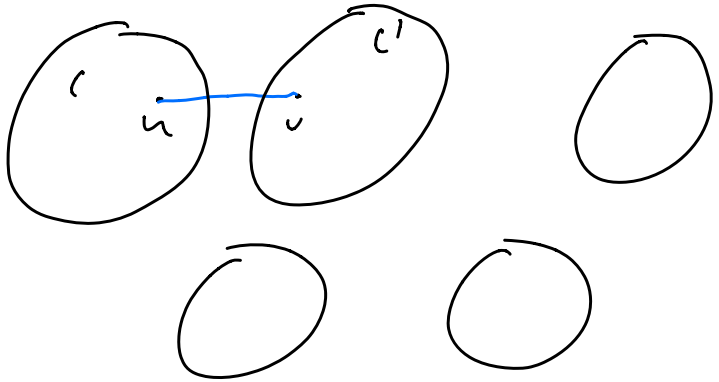

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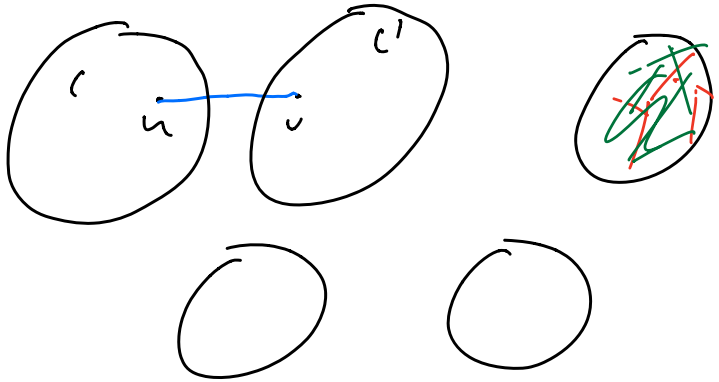

Consider cut $(C, \bar{C})$. Respects A, and $\{u, v\}$ light for it.
Main structural theorem $\implies$ $\{u, v\}$ safe for A

Running Time

Sorting edges: $O(m \log m) = O(m \log n)$

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- Finds:

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- Make-Sets: n
- Finds: $2m$

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Can we speak this up with data structures?

Union-Find! Connected components of A are disjoint sets.

- Make-Sets: n
- Finds: $2m$
- Unions:

Running Time

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Sorting dominates! $O(m \log n)$ total.