Lecture 19: Matroids and the Greedy Algorithm

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601.433/633 Introduction to Algorithms

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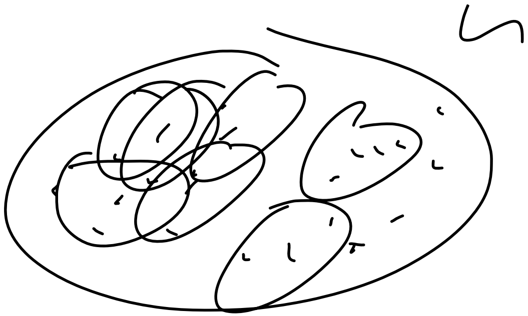
- Universe $\boldsymbol{U}$
- Collection $\mathcal{I} \subseteq \mathbf{2}^{\boldsymbol{U}}$ (so $\boldsymbol{I} \subseteq \boldsymbol{U}$ for all $\boldsymbol{I} \in \mathcal{I}$). Called *independent sets*
- Weights $\boldsymbol{w} : \boldsymbol{U} \to \mathbb{R}^+$



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Problem: find *max weight* independent set

$$w(S) = \sum_{e \in S} w(e)$$

MST as Weighted Set System

MST: weighted graph $\boldsymbol{G = (V, E, w)}$. Find MST.

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For any tree T:

$$w'(T) = \sum_{e \in T} w'(e) = \sum_{e \in T} (\bar{w} - w(e)) = \sum_{e \in T} \bar{w} - \sum_{e \in T} w(e) = (n-1)\bar{w} - w(T)$$

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So under weights w', max-weight IS $=$ max-weight forest $=$ max-weight spanning tree $=$ min-weight spanning tree (weights w)

- So finding max-weight forest $=$ finding min spanning tree.

Useful Properties of Forests

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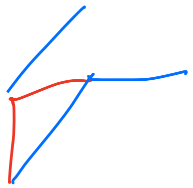

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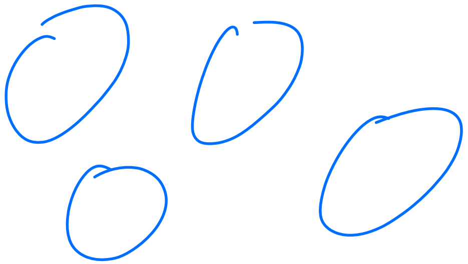

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Contradiction. $\square$

Matroids

Definition

$(U, \mathcal{I})$ is a *matroid* if the following three properties hold:

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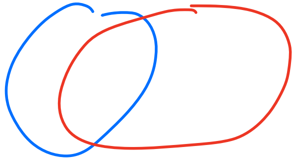

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Warmup: In any matroid, the maximal independent sets (called bases) have the same size (called the rank of the matroid).

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  - $\boldsymbol{U}$ a finite set of vectors in $\mathbb{R}^{\boldsymbol{d}}$
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  - Augmentation: if F_1 linearly independent, F_2 linearly independent, and $|F_2| > |F_1| \implies \dim(\mathrm{span}(F_1)) = |F_1| < |F_2| = \dim(\mathrm{span}(F_2))$

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Matroids: generalize both graph theory and linear algebra!

- Originally invented by Whitney as an attempt to generalize the concept of "linear independence"

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We'll assume we have independence oracle.

Greedy Algorithm

Kruskal, generalized to matroids (and max weight)!

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$\boldsymbol{F} = \varnothing$
Sort $\boldsymbol{U}$ by weight (largest to smallest)
For each $\boldsymbol{u} \in \boldsymbol{U}$ in sorted order $\{$
    If $\boldsymbol{F} \cup \{\boldsymbol{u}\} \in \mathcal{I}$, add $\boldsymbol{u}$ to $\boldsymbol{F}$
$\}$
Return $\boldsymbol{F}$

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- $\boldsymbol{F = \{f_1, f_2, \ldots, f_r\}}$, where $\boldsymbol{w(f_i) \geq w(f_{i+1})}$ for all $\boldsymbol{i}$ (order added by greedy)
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$$w(e_z) \geq w(e_j) > w(f_j)$$

Contradiction! Greedy would add e_z next, not f_j.

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So for hereditary set systems, matroids exactly characterize when the greedy algorithm works!

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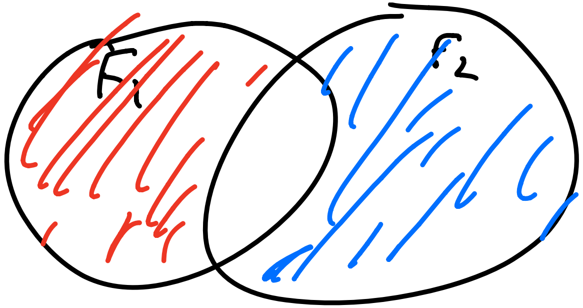

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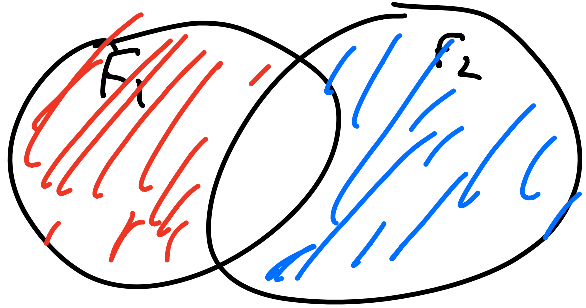

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2. $|F_2 \setminus F_1| \geq 1$
3. $|F_1 \setminus F_2| \geq 1$ (hereditary)

$\implies \exists \epsilon > 0$ such that $0 < (1 + \epsilon)|F_1 \setminus F_2| < |F_2 \setminus F_1|$

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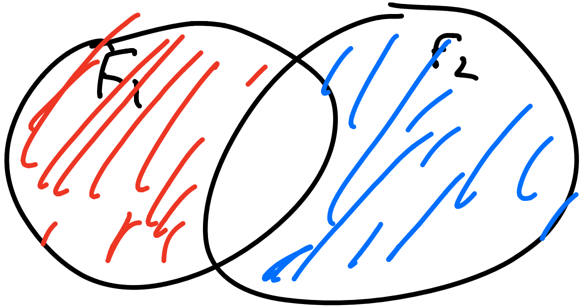


Easy facts:

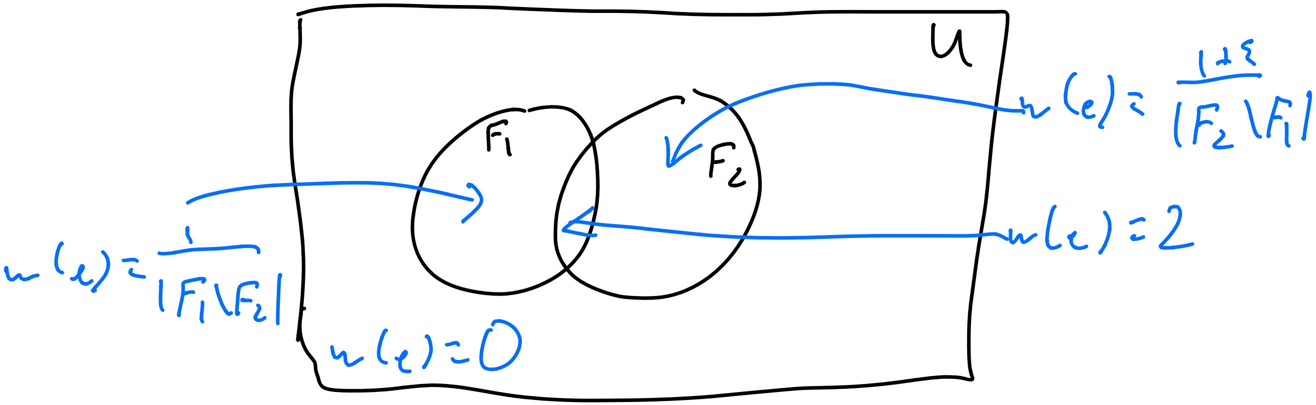
1. $|F_2 \setminus F_1| > |F_1 \setminus F_2|$
2. $|F_2 \setminus F_1| \geq 1$
3. $|F_1 \setminus F_2| \geq 1$ (hereditary)

$\implies \exists \epsilon > 0$ such that $0 < (1+\epsilon)|F_1 \setminus F_2| < |F_2 \setminus F_1|$

$$\implies \frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$$

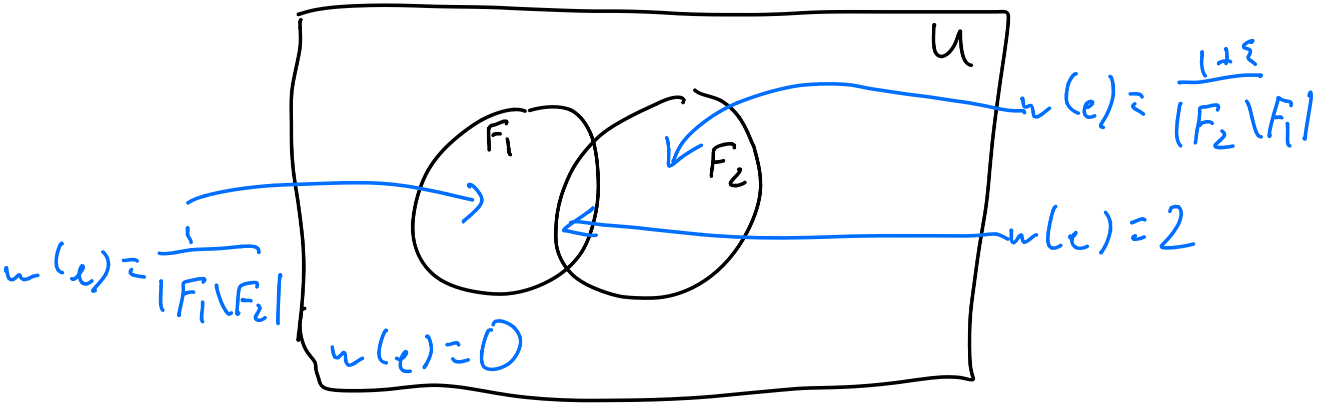
Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.



Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.

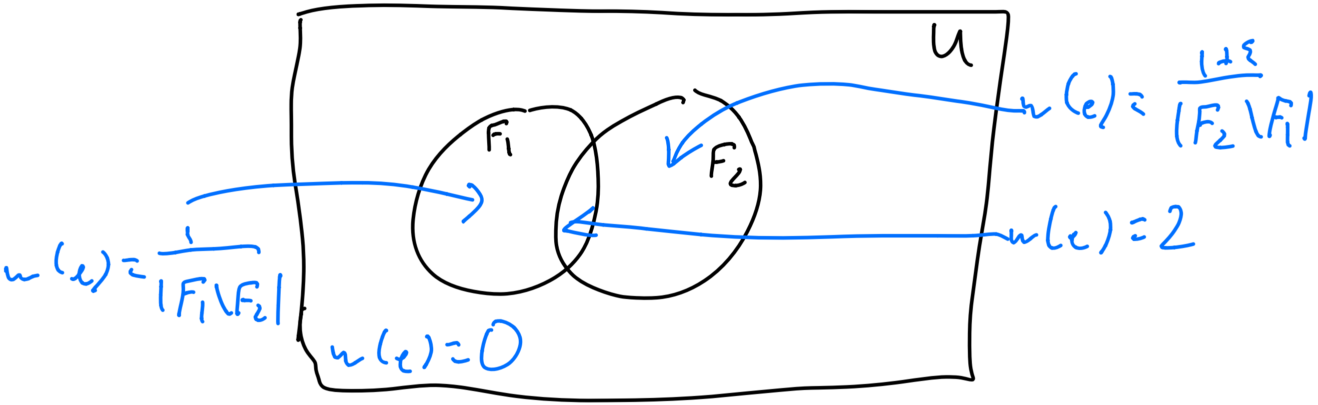


Greedy:

- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.



Greedy:

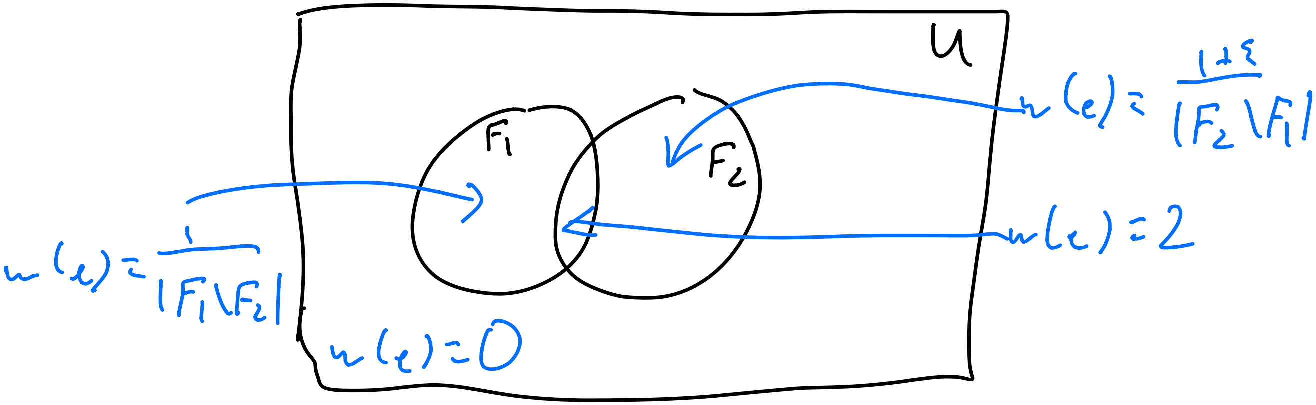
- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

$$\boldsymbol{w}(\text{greedy}) = 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|}$$

$$= 2|F_1 \cap F_2| + 1$$

Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.



Greedy:

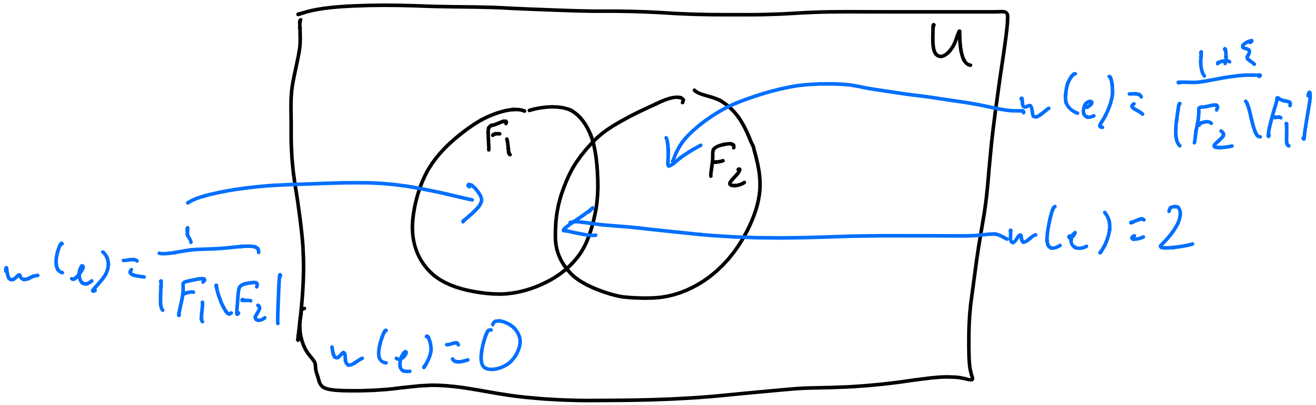
- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

$$w(\text{greedy}) = 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|} = 2|F_1 \cap F_2| + 1$$

$$w(F_2) = 2|F_1 \cap F_2| + |F_2 \setminus F_1| \frac{1+\epsilon}{|F_2 \setminus F_1|} = 2|F_1 \cap F_2| + 1 + \epsilon$$

Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.



Greedy:

- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

$$\begin{aligned} w(\text{greedy}) &= 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|} \\ &= 2|F_1 \cap F_2| + 1 \end{aligned}$$

$$\begin{aligned} w(F_2) &= 2|F_1 \cap F_2| + |F_2 \setminus F_1| \frac{1+\epsilon}{|F_2 \setminus F_1|} \\ &= 2|F_1 \cap F_2| + 1 + \epsilon \end{aligned}$$

Greedy not optimal: contradiction!