# Lecture 19: Matroids and the Greedy Algorithm

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601.433/633 Introduction to Algorithms
Slides by Michael Dinitz

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- Universe $\boldsymbol{U}$
- Collection $\mathcal{I} \subseteq \mathbf{2}^{\boldsymbol{U}}$ (so $\boldsymbol{I} \subseteq \boldsymbol{U}$ for all $\boldsymbol{I} \in \mathcal{I}$). Called *independent sets*
- Weights $\boldsymbol{w} : \boldsymbol{U} \to \mathbb{R}^+$

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Problem: find *max weight* independent set

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For any tree $T$:

$$w'(T) = \sum_{e \in T} w'(e) = \sum_{e \in T} (\bar{w} - w(e)) = \sum_{e \in T} \bar{w} - \sum_{e \in T} w(e) = (n-1)\bar{w} - w(T)$$

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So under weights $w'$, max-weight IS $=$ max-weight forest $=$ max-weight spanning tree $=$ min-weight spanning tree (weights $w$)

- So finding max-weight forest $=$ finding min spanning tree.

# Useful Properties of Forests

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## Proof Sketch that Forests have Augmentation Property.

Suppose false: no edge in $\boldsymbol{F_2} \setminus \boldsymbol{F_1}$ can be added to $\boldsymbol{F_1}$. Let $\boldsymbol{c_1}$ = # components in $\boldsymbol{F_1}$, $\boldsymbol{c_2}$ = # components in $\boldsymbol{F_2}$

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But $c_2 = n - |F_2| < n - |F_1| = c_1$.

Contradiction. $\square$

# Matroids

## Definition

$(U, \mathcal{I})$ is a *matroid* if the following three properties hold:

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Warmup: In any matroid, the maximal independent sets (called bases) have the same size (called the rank of the matroid).

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  - $\boldsymbol{U}$ a finite set of vectors in $\mathbb{R}^{\boldsymbol{d}}$
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  - $\varnothing$ linearly independent
  - If $F$ linearly independent and $F' \subseteq F$, then $F'$ linearly independent
  - Augmentation: if $F_1$ linearly independent, $F_2$ linearly independent, and $|F_2| > |F_1| \implies \dim(\text{span}(F_1)) = |F_1| < |F_2| = \dim(\text{span}(F_2))$

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Matroids: generalize both graph theory and linear algebra!

- Originally invented by Whitney as an attempt to generalize the concept of "linear independence"

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We'll assume we have independence oracle.

# Greedy Algorithm

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$\boldsymbol{F} = \varnothing$
Sort $\boldsymbol{U}$ by weight (largest to smallest)
For each $\boldsymbol{u} \in \boldsymbol{U}$ in sorted order $\{$
  If $\boldsymbol{F} \cup \{\boldsymbol{u}\} \in \mathcal{I}$, add $\boldsymbol{u}$ to $\boldsymbol{F}$
$\}$
Return $\boldsymbol{F}$

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**Theorem**

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Contradiction! Greedy would add $e_z$ next, not $f_j$.

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So for hereditary set systems, matroids exactly characterize when the greedy algorithm works!

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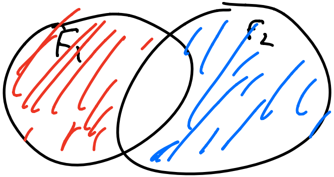


Easy facts:

1. $|F_2 \setminus F_1| > |F_1 \setminus F_2|$
2. $|F_2 \setminus F_1| \geq 1$
3. $|F_1 \setminus F_2| \geq 1$ (hereditary)

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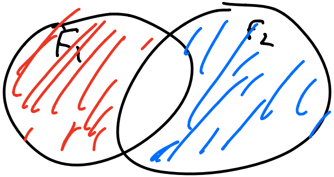


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$\implies \exists \epsilon > 0$ such that $0 < (1+\epsilon)|F_1 \setminus F_2| < |F_2 \setminus F_1|$

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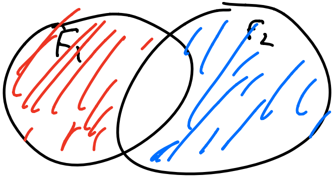


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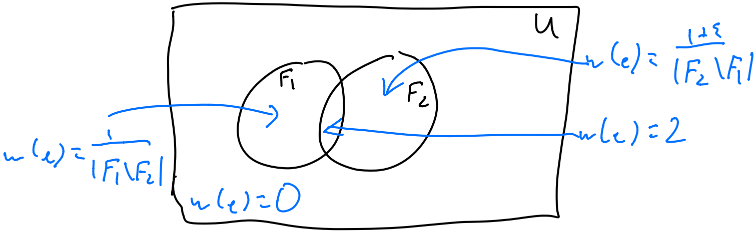
2. $|F_2 \setminus F_1| \geq 1$
3. $|F_1 \setminus F_2| \geq 1$ (hereditary)

$\implies \exists \epsilon > 0$ such that $0 < (1+\epsilon)|F_1 \setminus F_2| < |F_2 \setminus F_1|$

$$\implies \frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$$

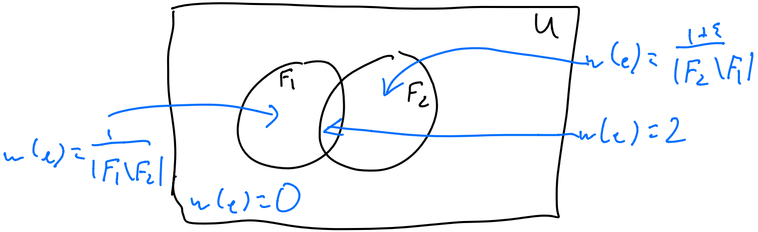
# Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.



# Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.

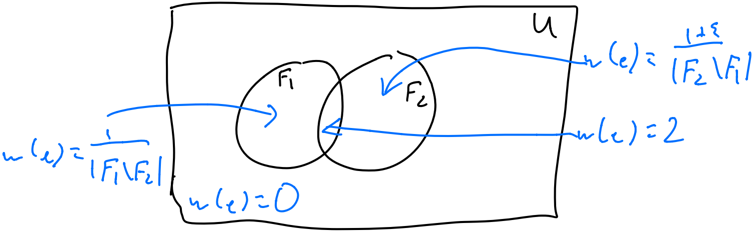



Greedy:

- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

# Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.




Greedy:

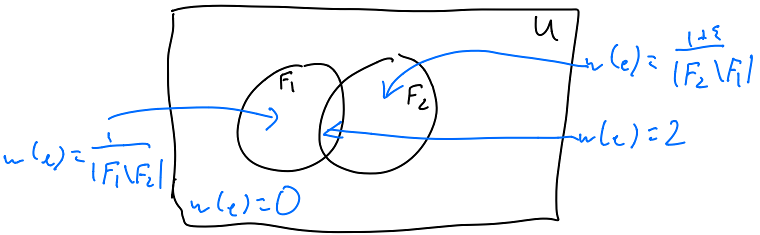
- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

$$w(\text{greedy}) = 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|}$$

$$= 2|F_1 \cap F_2| + 1$$

# Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.




Greedy:

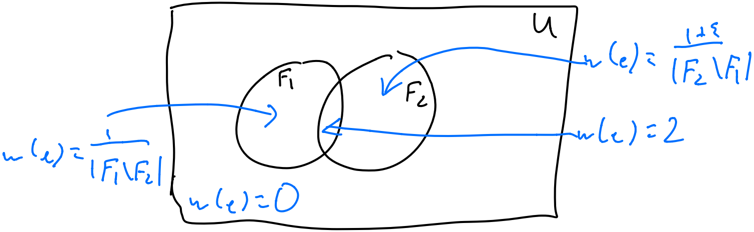
- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

$$w(\text{greedy}) = 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|} = 2|F_1 \cap F_2| + 1$$

$$w(F_2) = 2|F_1 \cap F_2| + |F_2 \setminus F_1| \frac{1+\epsilon}{|F_2 \setminus F_1|} = 2|F_1 \cap F_2| + 1 + \epsilon$$

# Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.




Greedy:

- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

$$\begin{aligned} w(\text{greedy}) &= 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|} \\ &= 2|F_1 \cap F_2| + 1 \end{aligned}$$

$$\begin{aligned} w(F_2) &= 2|F_1 \cap F_2| + |F_2 \setminus F_1| \frac{1+\epsilon}{|F_2 \setminus F_1|} \\ &= 2|F_1 \cap F_2| + 1 + \epsilon \end{aligned}$$

Greedy not optimal: contradiction!