Lecture 20: Max-Flow Min-Cut

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601.433/633 Introduction to Algorithms

Slides by Michael Dinitz

Introduction

Flow Network:

- Directed graph $G = (V, E)$
- *Capacities* $c : E \to \mathbb{R}_{\geq 0}$ (simplify notation: $c(x, y) = 0$ if $(x, y) \notin E$)
- Source $s \in V$, sink $t \in V$

Today: flows and cuts

- Flow: "sending stuff" from s to t
- Cut: separating t from s

Turn out to be very related!

Today: some algorithms but not efficient. Mostly structure. Better algorithms Tuesday.

Flows

Intuition: send “stuff” from $\boldsymbol{s}$ to $\boldsymbol{t}$

- Water in a city water system, traffic along roads, trains along tracks, ...

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$$\sum_{u:(u,v)\in E} f(u,v) = \sum_{u:(v,u)\in E} f(v,u)$$

for all $v \in V \setminus \{s,t\}$. This constraint is known as *flow conservation*.

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Value of flow $|f|$: "total amount of stuff sent from s to t"

$$|f| = \sum_{u:(s,u)\in E} f(s,u) - \sum_{u:(u,s)\in E} f(u,s) = \sum_{u:(u,t)\in E} f(u,t) - \sum_{u:(t,u)\in E} f(t,u)$$

Feasible Flows

Capacity constraints: $\mathbf{0} \leq \boldsymbol{f}(\boldsymbol{u}, \boldsymbol{v}) \leq \boldsymbol{c}(\boldsymbol{u}, \boldsymbol{v})$ for all $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{V} \times \boldsymbol{V}$

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Definitions:

- An (s,t)-flow satisfying capacity constraints is a *feasible* flow.
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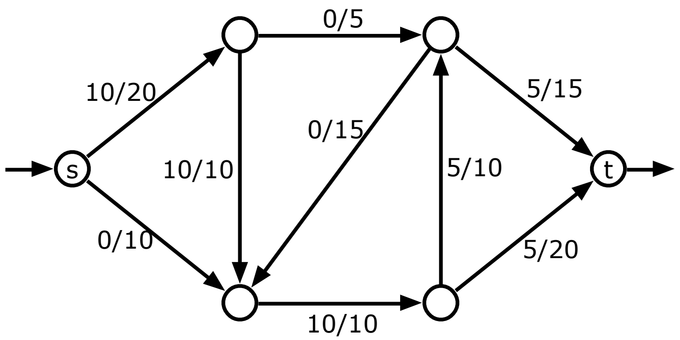

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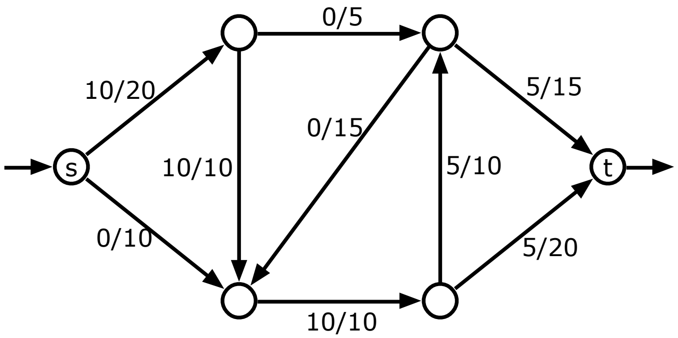

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Problem we'll talk about: find feasible flow of maximum value (max flow)

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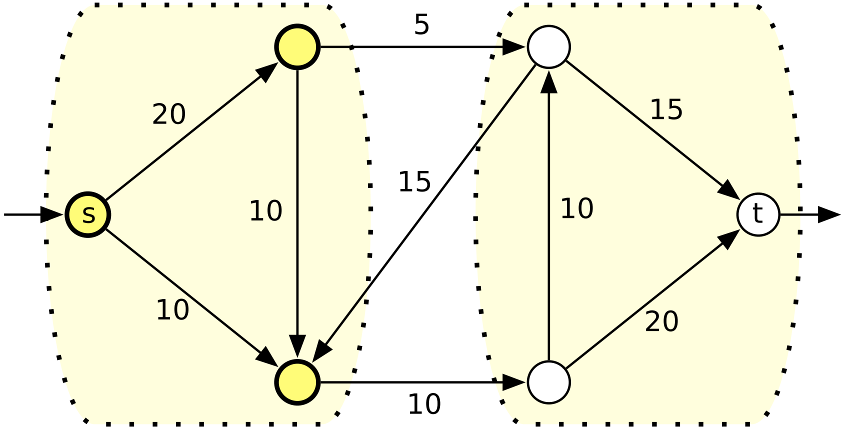

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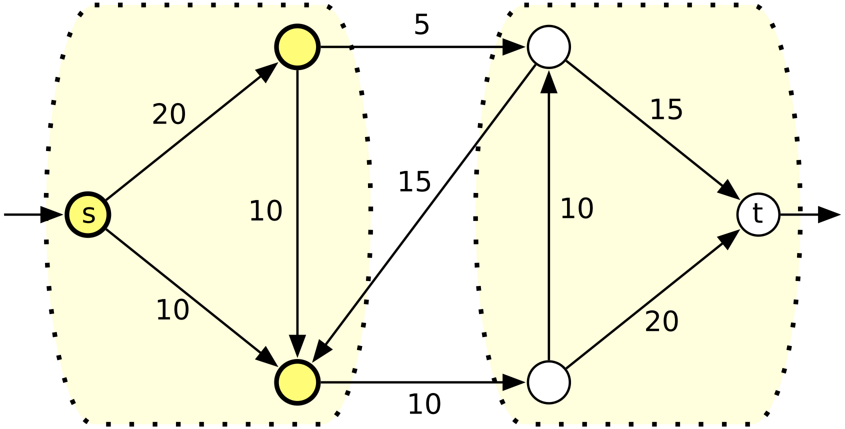

Problem we'll talk about: find (s, t)-cut of minimum capacity (min cut)

Warmup Theorem

Theorem

Let $\boldsymbol{f}$ be a feasible $(\boldsymbol{s}, \boldsymbol{t})$-flow, and let $(\boldsymbol{S}, \bar{\boldsymbol{S}})$ be an $(\boldsymbol{s}, \boldsymbol{t})$-cut. Then $|\boldsymbol{f}| \leq \boldsymbol{cap}(\boldsymbol{S}, \bar{\boldsymbol{S}})$.

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$$= \sum_{u \in S}\left(\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u)\right) \qquad \text{(flow conservation constraints)}$$

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&= \sum_{u \in S} \left(\sum_{v \in \bar{S}} f(u, v) - \sum_{v \in \bar{S}} f(v, u) \right) && \text{(remove terms which cancel)}
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&\le \sum_{u\in S}\sum_{v\in \bar{S}} f(u,v) && \text{(flow is nonnegative)}\\
&\le \sum_{u\in S}\sum_{v\in \bar{S}} c(u,v) = cap(S,\bar{S}) && \text{(flow is feasible)}
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Max-Flow Min-Cut

Corollary

If f avoids every $\bar{S} \to S$ edge and saturates every $S \to \bar{S}$ edge, then f is a maximum flow and $(S, \bar{S})$ is a minimum cut.

Max-Flow Min-Cut

Corollary

If $\boldsymbol{f}$ *avoids every* $\bar{\boldsymbol{S}} \to \boldsymbol{S}$ *edge and saturates every* $\boldsymbol{S} \to \bar{\boldsymbol{S}}$ *edge, then* $\boldsymbol{f}$ *is a maximum flow and* $(\boldsymbol{S}, \bar{\boldsymbol{S}})$ *is a minimum cut.*

Theorem (Max-Flow Min-Cut Theorem)

In any flow network, value of max $(\boldsymbol{s}, \boldsymbol{t})$*-flow* $=$ *capacity of min* $(\boldsymbol{s}, \boldsymbol{t})$*-cut.*

Max-Flow Min-Cut

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If $\boldsymbol{f}$ avoids every $\bar{\boldsymbol{S}} \to \boldsymbol{S}$ edge and saturates every $\boldsymbol{S} \to \bar{\boldsymbol{S}}$ edge, then $\boldsymbol{f}$ is a maximum flow and $(\boldsymbol{S}, \bar{\boldsymbol{S}})$ is a minimum cut.

Theorem (Max-Flow Min-Cut Theorem)

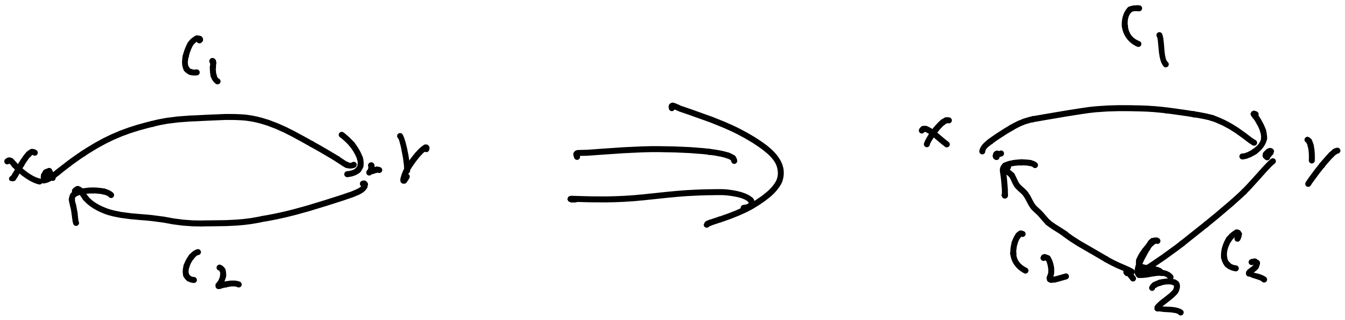
In any flow network, value of max $(\boldsymbol{s}, \boldsymbol{t})$-flow $=$ capacity of min $(\boldsymbol{s}, \boldsymbol{t})$-cut.

Spend rest of today proving this.

- Many different valid proofs.
- We'll see a classical proof which will naturally lead to algorithms for these problems.

One Direction

Cycles of length **2** will turn out to be annoying. Get rid of them.



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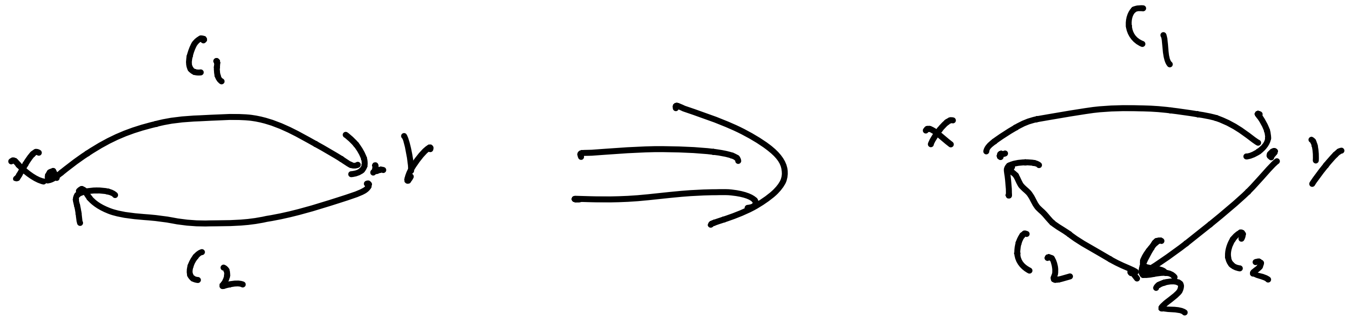

- Doesn't change max-flow or min-cut
- Increases #edges by constant factor, # nodes to original # edges.

Residual

Let f be feasible (s, t)-flow. Define *residual capacities*:

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases}$$

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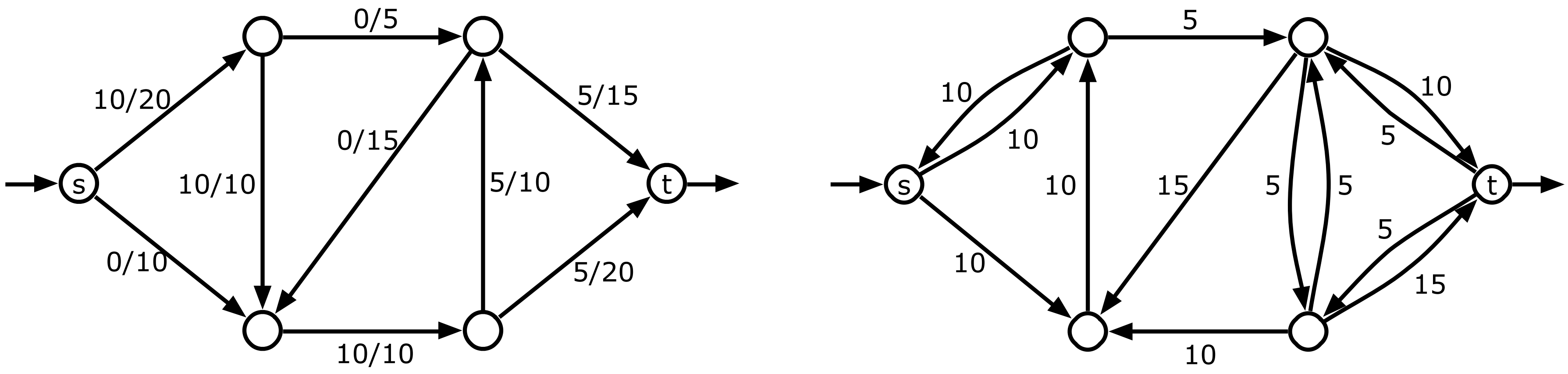
Residual Graph: $G_f = (V, E_f)$ where $(u,v) \in E_f$ if $c_f(u,v) > 0$.



A flow f in a weighted graph G and the corresponding residual graph G_f.

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Want to Show: There is a cut $(S, \bar{S})$ with $cap(S, \bar{S}) = |f|$.

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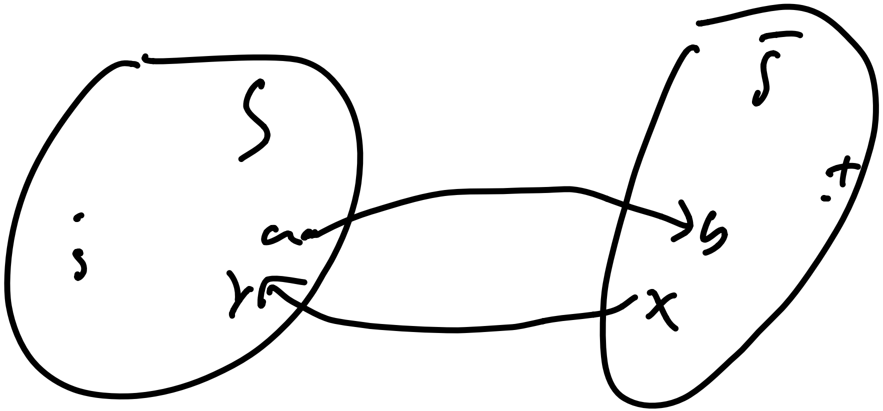

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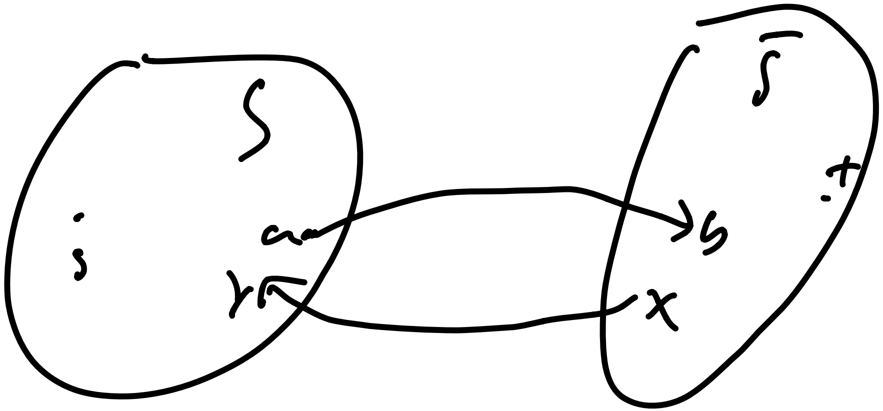

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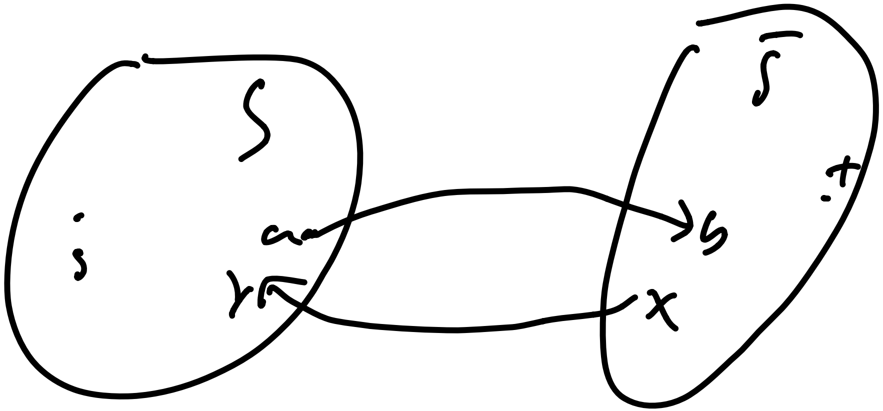

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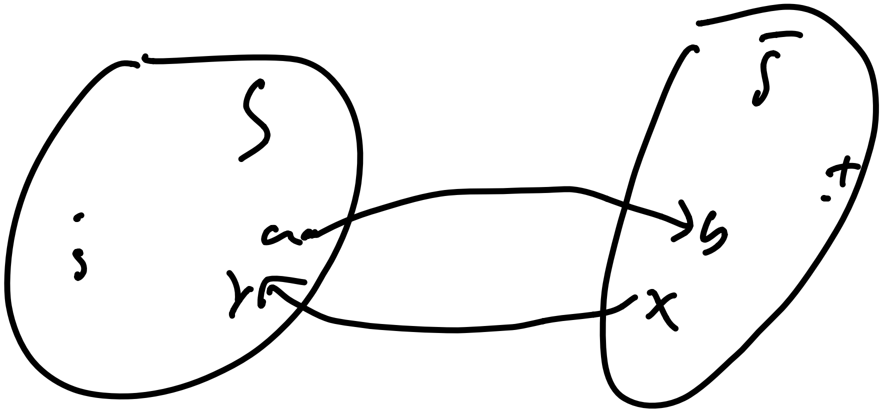

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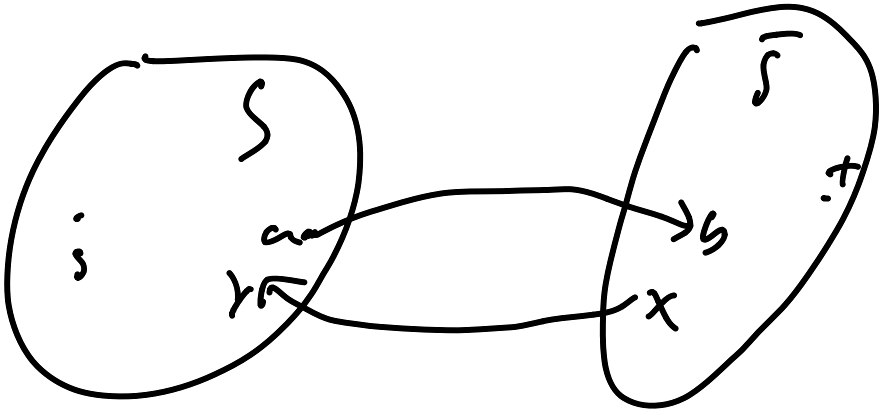

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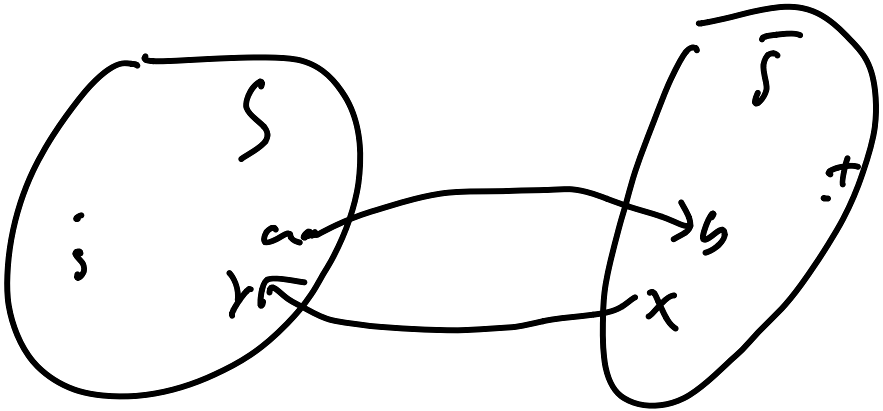

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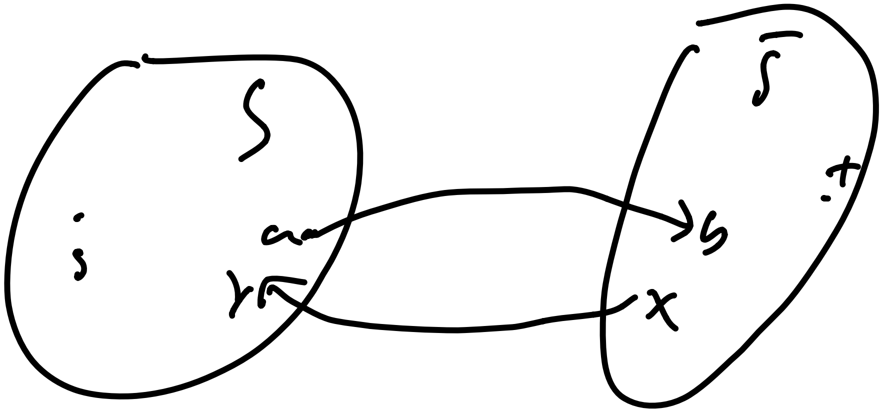

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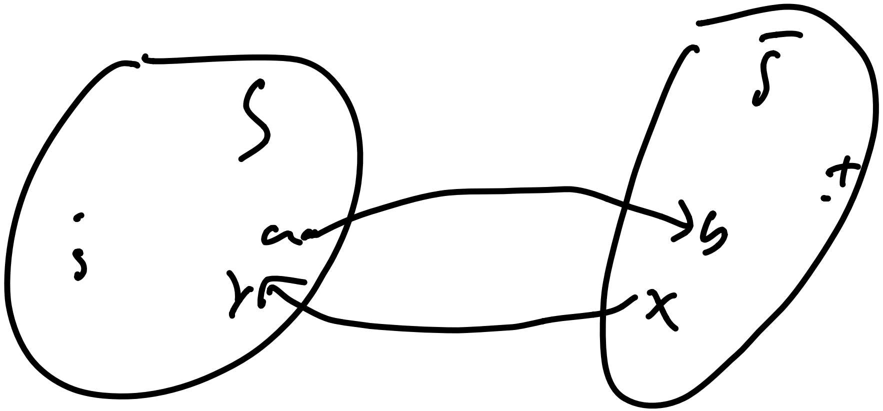

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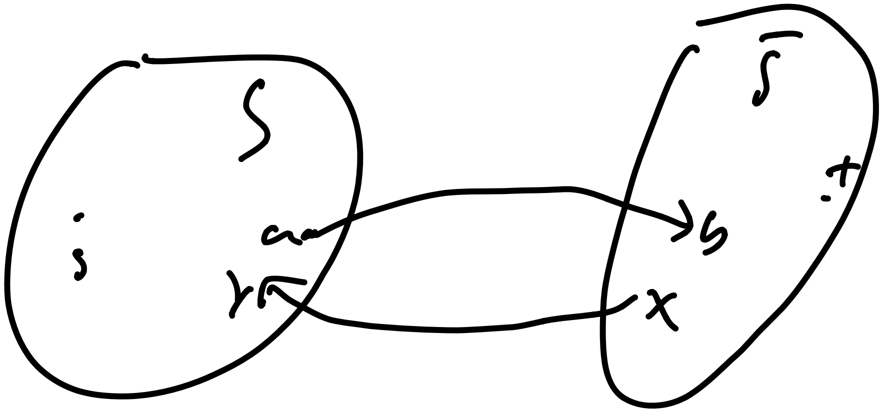

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f saturates $S \to \bar{S}$ edges, avoids $\bar{S} \to S$ edges $\implies cap(S,\bar{S}) = |f|$ by corollary

Case 2

Suppose $\exists$ an $\boldsymbol{s} \to \boldsymbol{t}$ path $\boldsymbol{P}$ in $\boldsymbol{G_f}$.

- Called an *augmenting path*

Idea: show that we can "push" more flow along $\boldsymbol{P}$, so $\boldsymbol{f}$ not a max flow. Contradiction, can't be in this case.

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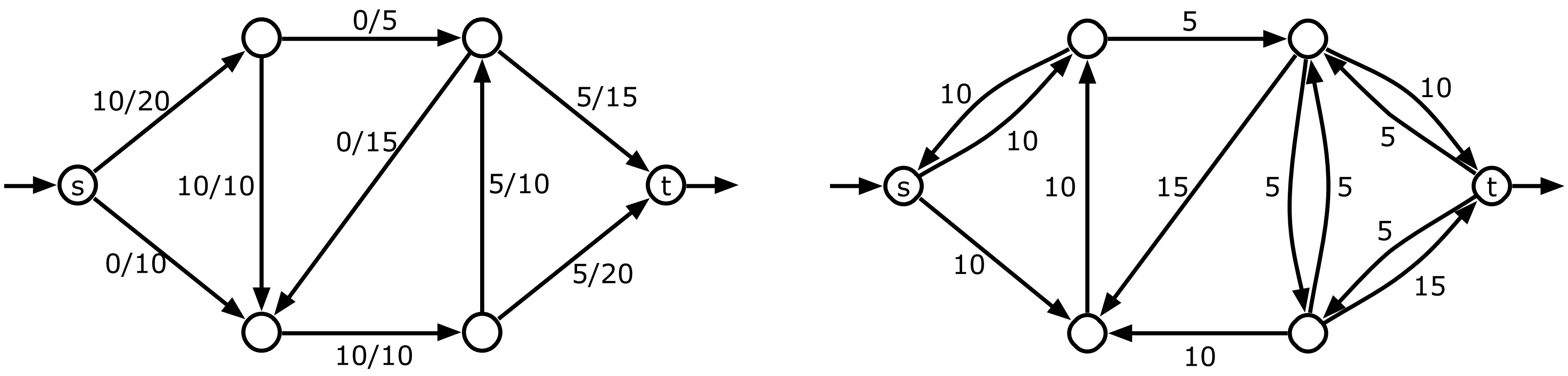
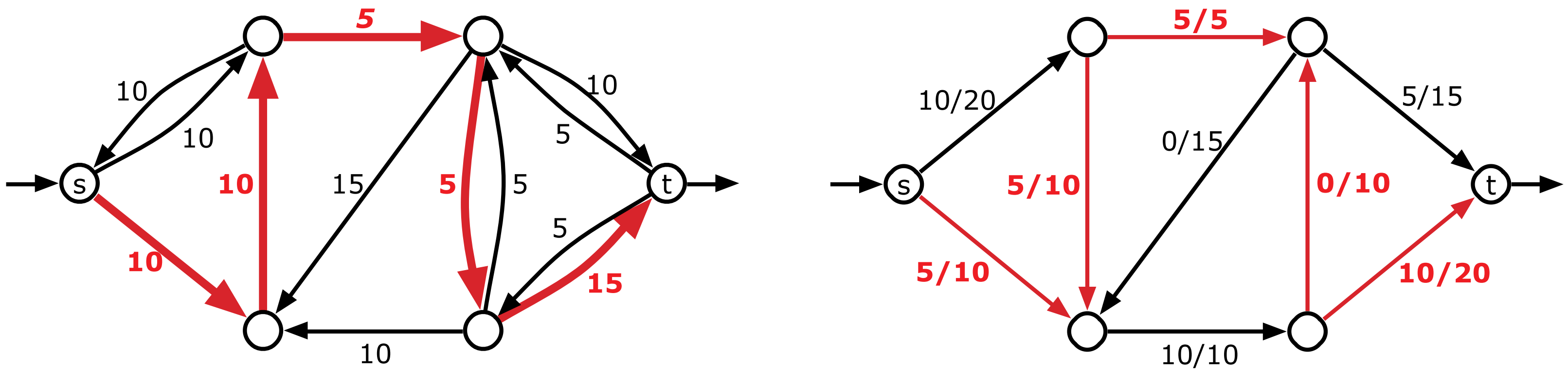
- Foreshadowing: augmenting path allows us to send more flow. Algorithm to increase flow!

Intuition



A flow f in a weighted graph G and the corresponding residual graph G_f.



An augmenting path in G_f with value $F = 5$ and the augmented flow f'.

Formalities

Let $\boldsymbol{P}$ be (simple) augmenting path in $\boldsymbol{G_f}$. Let $\boldsymbol{F} = \mathbf{min}_{\boldsymbol{e \in P}}\, \boldsymbol{c_f(e)}$.

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Let P be (simple) augmenting path in G_f. Let $F = \min_{e \in P} c_f(e)$.

Define new flow f': for all $(u, v) \in E$, let

$$f'(u,v) = \begin{cases} f(u,v) + F & \text{if } (u,v) \text{ in } P \\ f(u,v) - F & \text{if } (v,u) \text{ in } P \\ f(u,v) & \text{otherwise} \end{cases}$$

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Plan: prove (sketch) each subclaim individually

- $|f'| > |f|$
- f' an (s, t)-flow (flow conservation)
- f' feasible (obeys capacities)

$|f'| > |f|$

Consider first edge of P (out of s), say (s, v_1)

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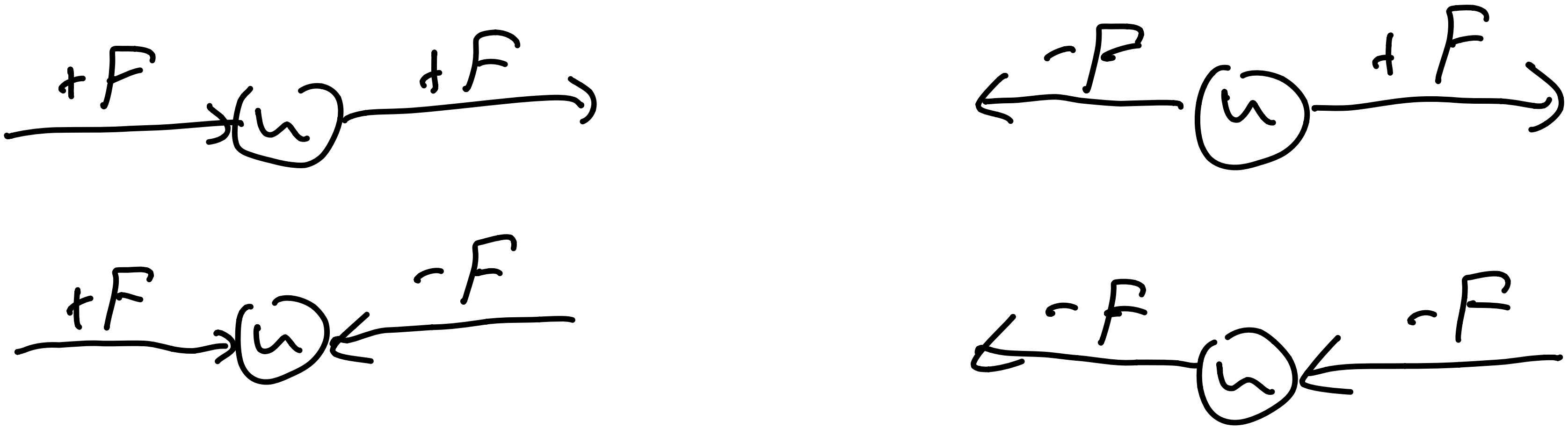

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Ford-Fulkerson Algorithm and Integrality

FF Algorithm

Obvious algorithm from previous proof: keep pushing flow!

$\boldsymbol{f} = \vec{\mathbf{0}}$
while($\exists \boldsymbol{s} \to \boldsymbol{t}$ path $\boldsymbol{P}$ in $\boldsymbol{G_f}$) {
    $\boldsymbol{F} = \mathbf{min}_{\boldsymbol{e} \in \boldsymbol{P}} \boldsymbol{c_f}(\boldsymbol{e})$
    Push $\boldsymbol{F}$ flow along $\boldsymbol{P}$ to get new flow $\boldsymbol{f}'$
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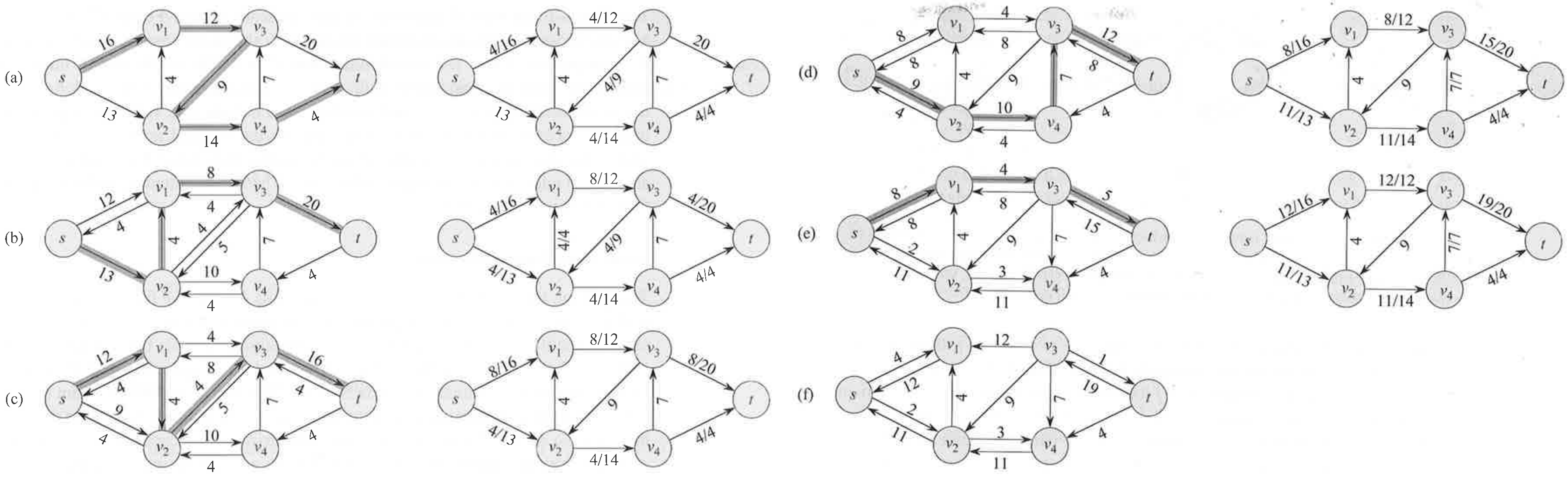
Correctness: directly from previous proof

Example



Integrality

Corollary

If all capacities are integers, then there is a max flow such that the flow through every edge is an integer

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Proof.

Induction on iterations of the Ford-Fulkerson algorithm: initially true, stays true $\implies$ true at end. $\square$

Running Time

Theorem

If all capacities are integers and the max flow value is F, Ford-Fulkerson takes time at most $O(F(m+n))$

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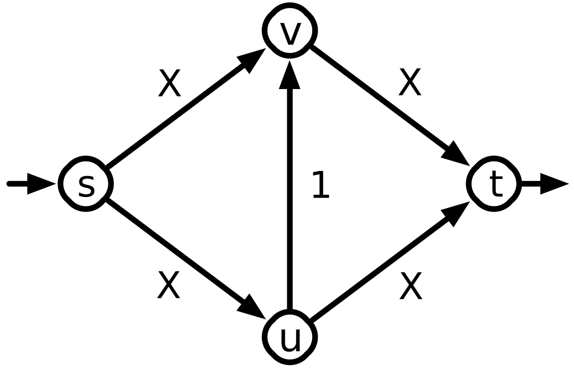

A bad example for the Ford-Fulkerson algorithm.

Running Time

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If all capacities are integers and the max flow value is $\boldsymbol{F}$, Ford-Fulkerson takes time at most $\boldsymbol{O(F(m+n))}$

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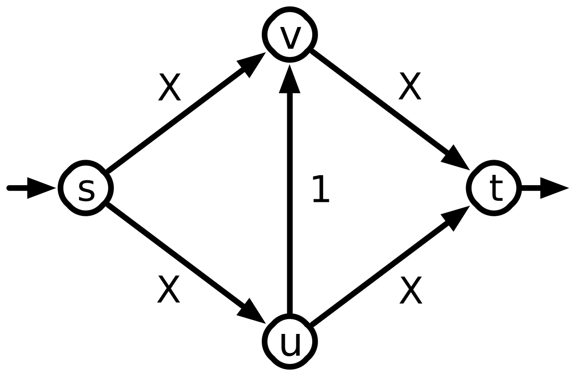

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Running time ≥ # iterations.

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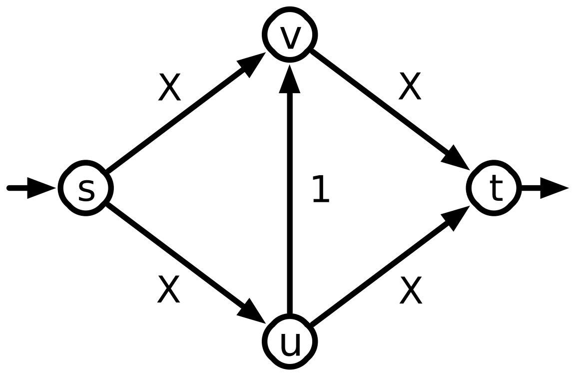

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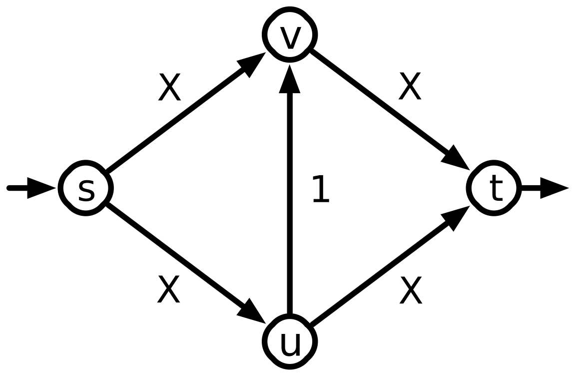

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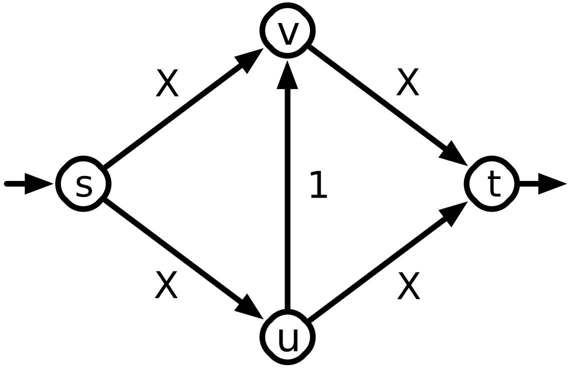

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$\implies$ Exponential time!