Lecture 21: Max-Flow II

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601.433/633 Introduction to Algorithms
Slides by Mike Dinitz

Introduction

Last time:

- Max-Flow = Min-Cut
- Can compute max flow and min cut using Ford-Fulkerson: while residual graph has an $s \to t$ path, push flow along it.
  - Corollary: if all capacities integers, max-flow is integral
  - If max-flow has value F, time $O(F(m+n))$ (if all capacities integers)
  - Exponential time!

Today:

- Important setting where FF is enough: max bipartite matching
- Two ways of making FF faster: Edmonds-Karp

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Proof.

Induction on iterations of the Ford-Fulkerson algorithm: initially true, stays true $\implies$ true at end. $\square$

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Theorem

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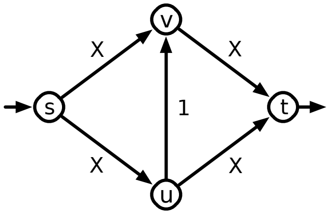
Finding path takes $O(m+n)$ time, increase flow by at least 1



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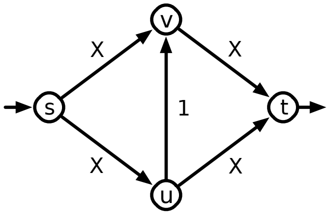

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Running time $\geq$ # iterations.

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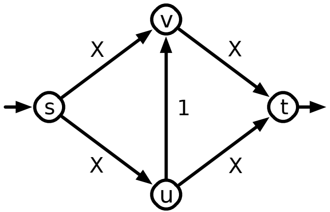

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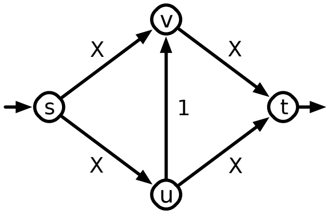

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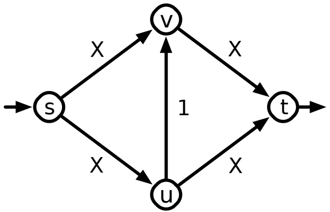

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Max Bipartite Matching

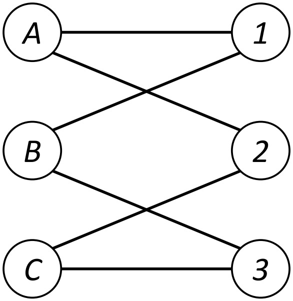
Setup

Definition

A graph $G = (V, E)$ is *bipartite* if V can be partitioned into two parts L, R such that every edge in E has one endpoint in L and one endpoint in R.

Definition

A *matching* is a subset $M \subseteq E$ such that $e \cap e' = \emptyset$ for all $e, e' \in M$ with $e \neq e'$ (no two edges share an endpoint)



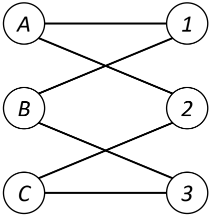
Bipartite Maximum Matching: Given bipartite graph $G = (V, E)$, find matching M maximizing $|M|$

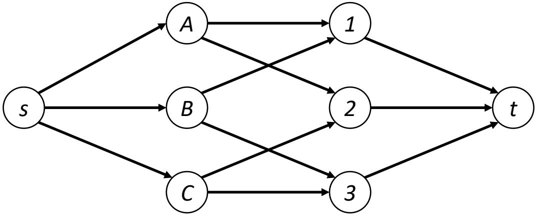
- Extremely important problem, doesn't seem to have much to do with flow!

Algorithm

Give all edges capacity $\mathbf{1}$
Direct all edges from $\boldsymbol{L}$ to $\boldsymbol{R}$
Add source $\boldsymbol{s}$ and sink $\boldsymbol{t}$
Add edges of capacity $\mathbf{1}$ from $\boldsymbol{s}$ to $\boldsymbol{L}$
Add edges of capacity $\mathbf{1}$ from $\boldsymbol{R}$ to $\boldsymbol{t}$

Run FF to get flow $\boldsymbol{f}$
Return $\boldsymbol{M} = \{\boldsymbol{e} \in \boldsymbol{L} \times \boldsymbol{R} : \boldsymbol{f}(\boldsymbol{e}) > \mathbf{0}\}$





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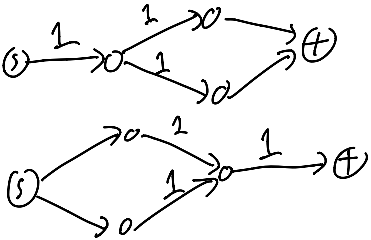

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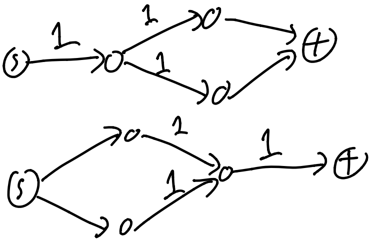

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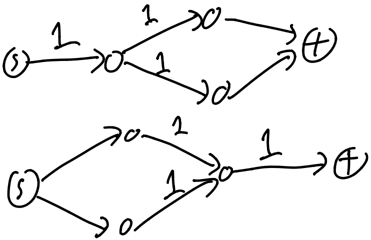

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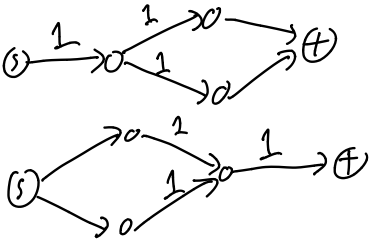

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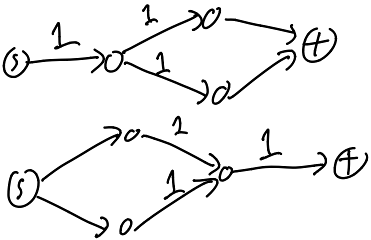

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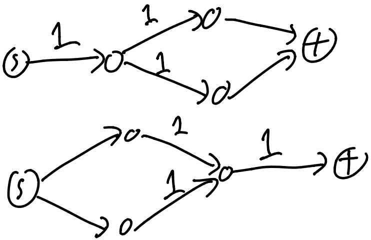

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- $|f'| = |M'| > |M| = |f|$

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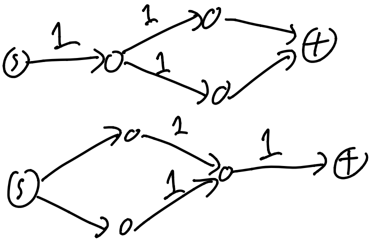

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- Contradiction

Running Time

Running Time:

- $O(n+m)$ to make new graph
- $|f| = |M| \leq n/2$ iterations of FF

$\Longrightarrow$ $O(n(m+n)) = O(mn)$ time (assuming $m \geq \Omega(n)$)

Exensions

Many extensions:

- Max-weight bipartite matching
- Min-cost perfect matching
- Matchings in general graphs
- ...

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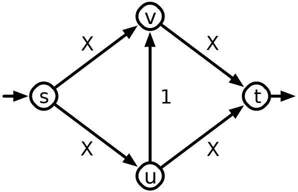
Still active area of study!

- Michael Dinitz, Sungjin Im, Thomas Lavastida, Benjamin Moseley, Sergei Vassilvitskii. *Faster Matchings via Learned Duals*. NeurIPS 2021.
- Michael Dinitz, George Li, Quanquan Liu, Felix Zhou. *Differentially Private Matchings*. Submitted, on arXiv.

Edmonds-Karp

Intuition

Bad example for Ford-Fulkerson:



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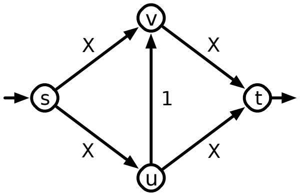

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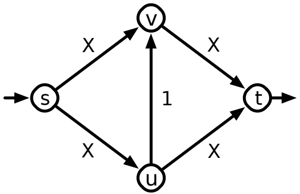

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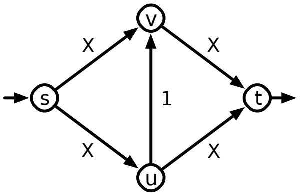

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(“widest” augmenting path)

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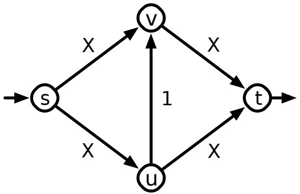

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Obvious path to pick:

$$\operatorname*{arg\,max}_{\text{augmenting paths } P} \min_{e \in P} c_f(e) \qquad (\text{“widest” augmenting path})$$

Less obvious path to pick:

$$\operatorname*{arg\,min}_{\text{augmenting paths } P} |P| \qquad (\text{augmenting path with fewest edges})$$

Edmonds-Karp

Use Ford-Fulkerson, but pick *shortest* augmenting path (unweighted)

- Ignore capacities, just find augmenting path with fewest hops!
- Easy to compute with BFS in $O(m + n)$ time.
- Correct, since just FF with particular path choice.

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Theorem

Edmonds-Karp has at most $O(mn)$ iterations, so at most $O(m^2 n)$ running time (if $m \geq n$)

Proof (sketch) of Edmonds-Karp

Idea: prove that distance from s to t (unweighted) goes up by at least one every $\leq m$ iterations.

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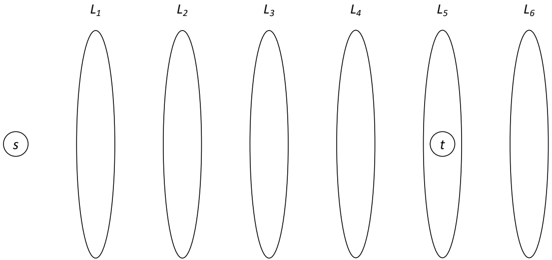
- Distance initially $\geq 1 \implies$ distance $> n$ after at most mn iterations
- Only distance larger than n is ∞: no $s \to t$ path

$\implies$ Terminates after at most mn iterations.

Proof (sketch) of Edmonds-Karp (continued)

Suppose $\boldsymbol{s} \to \boldsymbol{t}$ distance is $\boldsymbol{d}$.
"Lay out" residual graph in levels by BFS (distance from $\boldsymbol{s}$)



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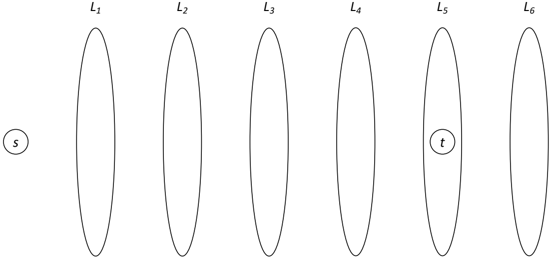

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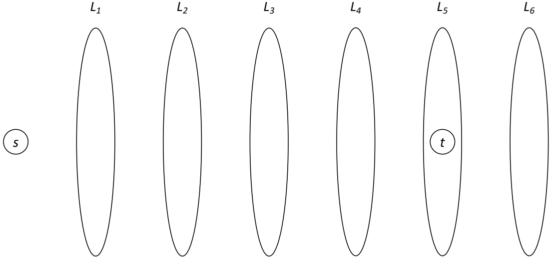

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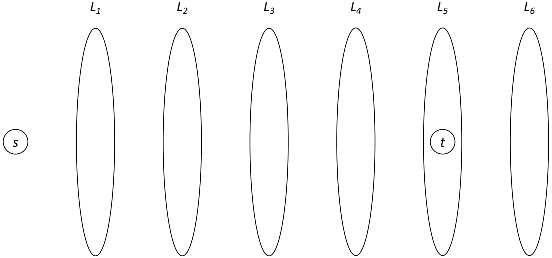

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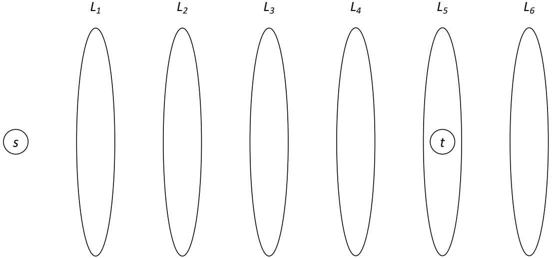

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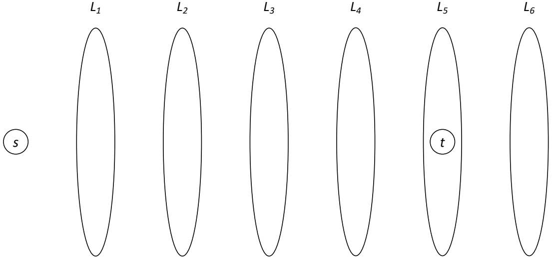

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So after m iterations (same layout): no path using only forward edges $\implies$ distance larger than d!

Finishing Edmonds-Karp

So at most mn iterations. Each iteration unweighted shortest path: BFS, time $O(m + n)$

Finishing Edmonds-Karp

So at most $\boldsymbol{mn}$ iterations. Each iteration unweighted shortest path: BFS, time $\boldsymbol{O(m+n)}$

Total time: $\boldsymbol{O(mn(m+n)) = O(m^2 n)}$. Independent of $\boldsymbol{F}$!

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Does this imply at most m iterations?

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- $i = 0$: F
- $I = 1$: Sent at least F/m, so at most $F - F/m = F(1 - 1/m)$ remaining

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Super useful inequality: $1 + x \leq e^x$ for all $x \in \mathbb{R}$
$\implies$ If $i > m \ln F$, amount remaining to be sent at most

$$F(1 - 1/m)^i < F(1 - 1/m)^{m \ln F} \leq F(e^{-1/m})^{m \ln F} = F \cdot e^{-\ln F} = 1$$

But all capacities integers, so must be finished!

Finishing up

Modified version of Dijkstra: find widest path in $O(m \log n)$ time

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We saw earlier how to get running time independent of F (a *strongly* polynomial-time algorithm running in $O(m^2 n)$).

Extensions

Many better algorithms for max-flow: *blocking flows* (Dinitz's algorithm (not that Dinitz)), *push-relabel* algorithms, etc.

- CLRS has a few of these.
- State of the art:
  - Strongly polynomial: $\boldsymbol{O(mn)}$. Orlin [2013] & King, Rao, Tarjan [1994]
  - Weakly Polynomial: $\boldsymbol{O(m^{1+o(1)} \log U)}$ (where $\boldsymbol{U}$ is maximum capacity). Chen, Kyng, Liu, Peng, Gutenberg and Sachdeva [2022]

Many other variants of flows, some of which are just $\boldsymbol{s - t}$ max flow in disguise!

- Min-Cost Max-Flow: every edge also has a cost. Find minimum cost max-flow. Can be solved with just normal max flow!