Lecture 22: Linear Programming

Michael Dinitz

November 13, 2025
601.433/633 Introduction to Algorithms

Introduction

Today: What, why, and juste a taste of how

- ▶ Entire course on linear programming over in AMS. Super important topic!
- ▶ Fast algorithms in theory and in practice.

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- Entire course on linear programming over in AMS. Super important topic!
- Fast algorithms in theory and in practice.

Why: Even more general than max-flow, can still be solved in polynomial time!

- Max flow important in its own right, but also because it can be used to solve many other things (max bipartite matching)
- Linear programming: important in its own right, but also even more general than max-flow.
- Can model many, many problems!

Example: Planning Your Week

168 hours in a week. How much time to spend:

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- ▸ $E \geq 56$ (at least 8 hours/day sleep, shower, etc.)
- ▸ $P + E \geq 70$ (need to stay sane)
- ▸ $S \geq 60$ (to pass your classes)
- ▸ $2S + E - 3P \geq 150$ (too much partying requires studying or sleep)

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- Yes! $S = 80$, $P = 20$, $E = 68$

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Question: Suppose "happiness" is $2P + 3E$. Can we find a feasible solution maximizing this?

Linear Programming

Input (a "linear program"):

- n variables $x_1, \ldots, x_n$ (take values in $\mathbb{R}$)
- m *non-strict linear inequalities* in these variables (constraints)
  - E.g.: $3x_1 + 4x_2 \leq 6$, $\quad 0 \leq x_1 \leq 3$ $\quad x_2 - 3x_3 + 2x_7 = 17$
  - Not allowed (examples): $x_2 x_3 \geq 5$, $\quad x_4 < 2$, $\quad x_5 + \log x_2 \geq 4$
- Possibly a *linear* objective function
  - $\max 2x_3 - 4x_5$, $\quad \min \frac{5}{2} x_4 + x_2$, $\quad \ldots$

Goals:

- Feasibility: Find values for x's that satisfy all constraints
- Optimization: Find feasible solutions maximizing/minimizing objective function

Both achievable in polynomial time, reasonably fast!

Planning your week as an LP

Variables: P, E, S

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& P + E \geq 70 \\
& P + S + E = 168 \\
& P \geq 0 \\
& S \geq 0 \\
& E \geq 0
\end{aligned}$$

Planning your week as an LP

Variables: P, E, S

$$\max \quad 2P + E$$
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$$S \geq 60$$
$$2S + E - 3P \geq 150$$
$$P + E \geq 70$$
$$P + S + E = 168$$
$$P \geq 0$$
$$S \geq 0$$
$$E \geq 0$$

When using an LP to model your problem, need to be sure that *all* aspects of your problem included!

Operations Research-style Example

Four different manufacturing plants for making cars:

	labor	materials	pollution
Plant 1	2	3	15
Plant 2	3	4	10
Plant 3	4	5	9
Plant 4	5	6	7

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- Need to produce at least **400** cars at plant 3 (labor agreement)
- Have **3300** total hours of labor, **4000** units of material
- Environmental law: produce at most **12000** pollution
- Make as many cars as possible

OR example as an LP

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Variables: x_i = # cars produced at plant i, for $i \in \{1, 2, 3, 4\}$

Objective: max $x_1 + x_2 + x_3 + x_4$

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Constraints:

$$x_3 \geq 400$$

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Constraints:

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$$2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 3300$$

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$$15x_1 + 10x_2 + 9x_3 + 7x_4 \leq 12000$$
$$x_i \geq 0 \qquad \forall i \in \{1, 2, 3, 4\}$$

Max Flow as LP
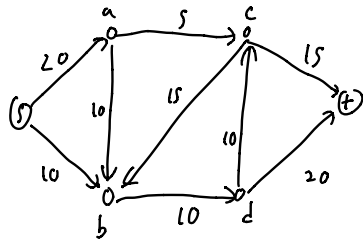
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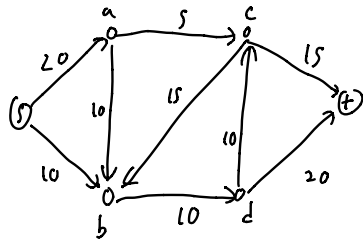
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Variables: $f(e)$ for all $e \in E$
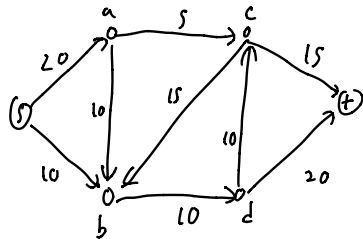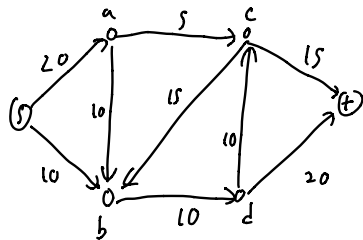
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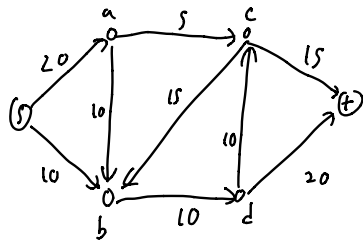
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Objective: $\max \sum_v f(s, v) - \sum_v f(v, s)$

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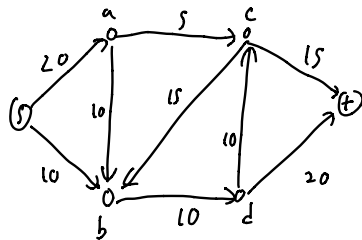**Objective:** $\max \sum_v f(s, v) - \sum_v f(v, s)$

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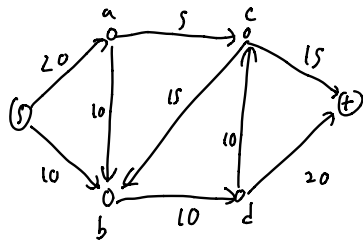
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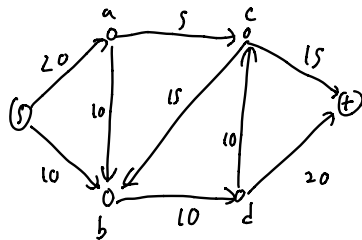
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So can solve max-flow and min-cut (slower) by using generic LP solver

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Generalization of max-flow with
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- Directed graph $G = (V, E)$
- Capacities $c : E \to \mathbb{R}_{\geq 0}$
- k source-sink pairs $\{(s_i, t_i)\}_{i \in [k]}$

Goal: send flow of commodity i from s_i to t_i, max total flow sent across all commodities

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Concurrent Flow

Multicommodity flow, but:

- Also given *demands*
  $d : [k] \to \mathbb{R}_{\geq 0}$
- Question: Is there a
  multicommodity flow
  that sends at least $d(i)$
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Maximum Concurrent Flow

If answer is no: how much do we need to scale down demands so that there is a multicommodity flow?

Maximum Concurrent Flow

Variables:

▸ $f_i(e)$ for all $e \in E$ and for all $i \in [k]$.

▸ λ

Objective: $\max \lambda$

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$$\sum_{i=1}^k f_i(e) \le c(e) \qquad \forall e \in E$$

$$f_i(e) \ge 0 \qquad \forall e \in E, \ \forall i \in [k]$$

$$\sum_v f_i(s_i, v) - \sum_v f_i(v, s_i) \ge \lambda d(i) \qquad \forall i \in [k]$$

Shortest $s - t$ path

Very surprising LP!
Variables: d_v for all $v \in V$: shortest-path distance from s to v

$$\max \quad d_t$$
$$\text{subject to} \quad d_s = 0$$
$$d_v \leq d_u + \ell(u, v) \qquad \forall (u, v) \in E$$

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Correctness Theorem: Let $\vec{d}^*$ denote the optimal LP solution. Then $d_t^* = d(s, t)$
Proof Sketch: $\ge$: Let $d_v = d(s, v)$ for all $v \in V$. Feasible $\implies d_t^* \ge d_t = d(s, t)$.

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Proof Sketch: $\geq$: Let $d_v = d(s, v)$ for all $v \in V$. Feasible $\implies d_t^* \geq d_t = d(s, t)$.

$\leq$: Let $P = (s = v_0, v_1, \ldots, v_k = t)$ be shortest $s \to t$ path.
Prove by induction: $d_{v_i}^* \leq d(s, v_i)$ for all i

Shortest $s - t$ path

Very surprising LP!
Variables: d_v for all $v \in V$: shortest-path distance from s to v

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$$\text{subject to} \quad d_s = 0$$
$$d_v \leq d_u + \ell(u, v) \qquad\qquad \forall (u, v) \in E$$

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Inductive step: $d_{v_i}^* \leq d_{v_{i-1}}^* + \ell(v_{i-1}, v_i) \leq d(s, v_{i-1}) + \ell(v_{i-1}, v_i) = d(s, v_i)$

Algorithms for LPs

Geometry

To get intuition: think of LPs *geometrically*

- Space: $\mathbb{R}^n$ (one dimension per variable
- Linear constraint: halfspace (one side of a hyperplane)
- Feasible region: intersection of halfspaces. *Convex Polytope* (usually just called a *polytope*)

Geometry

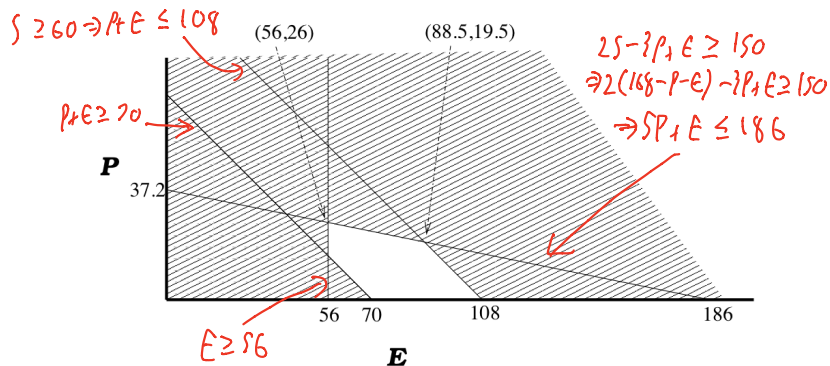To get intuition: think of LPs *geometrically*

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Example: planning your week

- ▸ **3** variables S, P, E so $\mathbb{R}^3$
- ▸ But $S + P + E = 168 \implies$
  $S = 168 - P - E$
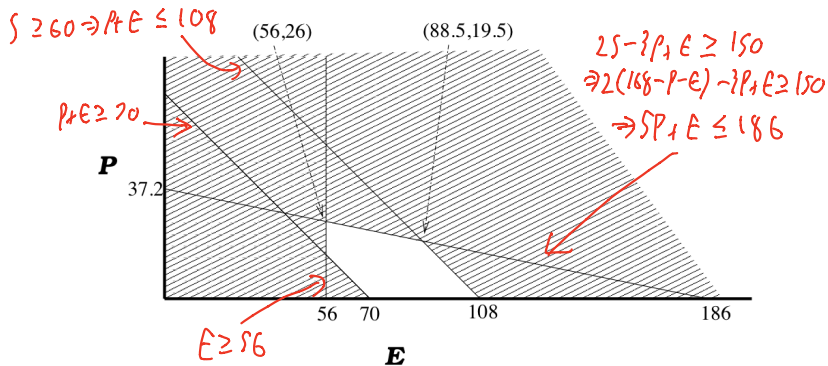- ▸ Make this substitution, get
  $\mathbb{R}^2$
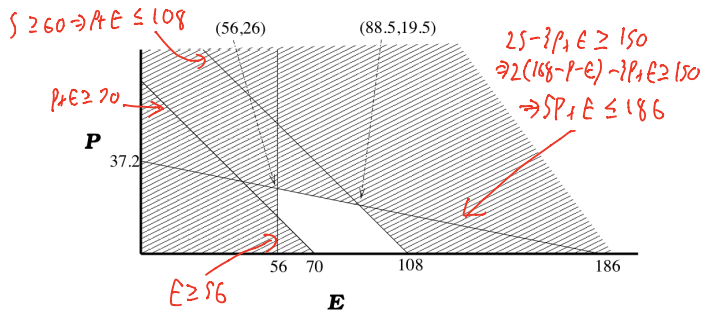
Geometry (cont'd)

Objective: feasible solution "furthest" along specified direction

- **max P**: $(56, 26)$
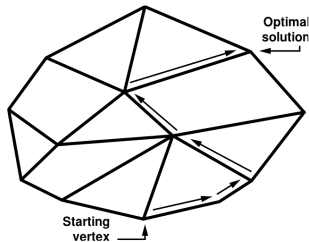- **max $2P + E$**: $(88.5, 19.5)$
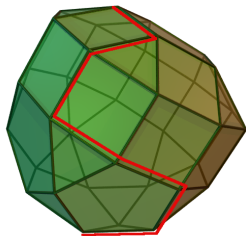
Geometry (cont'd)

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- **max $2P + E$**: $(88.5, 19.5)$

Main theorem: optimal solution is always at a "corner" (also called a "vertex")

Simplex Algorithm [Dantzig 1940's]

Initialize $\vec{x}$ to an arbitrary corner
while(a neighboring corner $\vec{x}'$ of $\vec{x}$ has better objective value) {
    $\vec{x} \leftarrow \vec{x}'$
}
return $\vec{x}$

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Theorem: Simplex returns the optimal solution.

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- ▶ Slow in theory
- ▶ Fast in practice!
  - ▶ Much of AMS LP course really about simplex: traditionally favorite algorithm of people who want to actually solve LPs
- ▶ Some theory to explain discrepancy ("smoothed analysis")

Ellipsoid Algorithm [Khachiyan 1980]

First polytime algorithm!
Designed to just solve feasibility question $\implies$ can also solve optimization

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First polytime algorithm!
Designed to just solve feasibility question $\implies$ can also solve optimization

- ▶ Start with ellipsoid E containing feasible region P (if it exists)
- ▶ Let x be center of E
- ▶ While(x not feasible)
  - ▶ Find a hyperplane H through x such that all of P on one side
  - ▶ Let E' be the half-ellipsoid of E defined by H
  - ▶ Find a new ellipsoid $\hat{E}$ containing E' so that $vol(\hat{E}) \leq \left(1 - \frac{1}{n}\right) vol(E)$
  - ▶ Let $E = \hat{E}$ and let x be center of $\hat{E}$

Analysis

Extremely complicated!

Geometry of ellipsoids: can always find an ellipsoid containing a half-ellipsoid with at most $(1 - 1/n)$ of the volume of the original

- Using inequality from last time: after n iterations, volume drops by $\left(1 - \frac{1}{n}\right)^n \leq 1/e$ factor
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$\implies$ Polynomial time!

In practice: horrible.

Interior Point Methods (Karmarkar's Algorithm)

Fast in both theory and practice!