

Lecture 23: NP-Completeness I

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601.433/633 Introduction to Algorithms

Introduction

Last few weeks: slower and slower algorithms for harder and harder problems

- ▶ From $O(m + n)$ time algorithms for BFS/DFS/topological sort/SCCs, to $O(m^2n)$ for max flow
- ▶ Today: start of two lectures on NP-completeness.
 - ▶ The (or at least a) line between tractability and intractability

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Definition

An algorithm runs in *polynomial time* if its (worst-case) running time is $O(n^c)$ for some constant $c \geq 0$, where n is the size of the input.

Think of polynomial time as “fast”, super-polynomial time as “slow”

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Question: When do polynomial-time algorithms exist?

Decision Problems

Definition

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- ▶ Max-Flow: Input is $\mathbf{G} = (\mathbf{V}, \mathbf{E}), \mathbf{c} : \mathbf{E} \rightarrow \mathbb{R}_{\geq 0}, \mathbf{s}, \mathbf{t} \in \mathbf{V}, \mathbf{k} \in \mathbb{R}^+$. Output YES if there is an $(\mathbf{s}, \mathbf{t})$ -flow of value at least $\mathbf{k}$, otherwise output NO.
- ▶ Shortest $\mathbf{s} - \mathbf{t}$ path: Input is $\mathbf{G} = (\mathbf{V}, \mathbf{E}), \ell : \mathbf{E} \rightarrow \mathbb{R}, \mathbf{s}, \mathbf{t} \in \mathbf{V}, \mathbf{k} \in \mathbb{R}$. Output YES if $d(\mathbf{s}, \mathbf{t}) \leq \mathbf{k}$, otherwise output NO.

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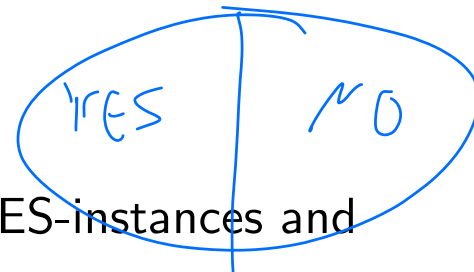
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- ▶ Max-Flow: Input is $G = (V, E), c : E \rightarrow \mathbb{R}_{\geq 0}, s, t \in V, k \in \mathbb{R}^+$. Output YES if there is an (s, t) -flow of value at least k , otherwise output NO.
- ▶ Shortest $s - t$ path: Input is $G = (V, E), \ell : E \rightarrow \mathbb{R}, s, t \in V, k \in \mathbb{R}$. Output YES if $d(s, t) \leq k$, otherwise output NO.

Some problems naturally decision, others naturally optimization, but can turn any optimization problem into a decision problem.

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Note: Can divide instances (inputs) of any decision problem into YES-instances and NO-instances



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Question: Are all decision problems in P ?

Answer: No!

- ▶ By *time hierarchy theorem* there are problems that require super-polynomial time!
- ▶ Undecidability: there are problems which cannot be solved by *any* algorithm at all!

Verification

Different Setting: If *in addition* to the input we're given a purported solution, can we check that this solution is valid/feasible (in polynomial time)?

- ▶ Max-Flow: given $f : E \rightarrow \mathbb{R}_{\geq 0}$, check that value $\geq k$, flow conservation at all nodes other than s, t , and capacity constraints obeyed

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Definition (3-Coloring)

Input: Undirected graph $G = (V, E)$

Output: YES if $\exists$ coloring $f : V \rightarrow \{R, G, B\}$ such that $f(u) \neq f(v)$ for all $\{u, v\} \in E$. NO otherwise

Verification

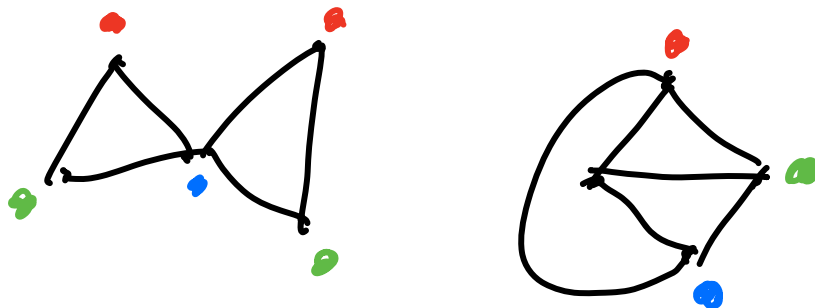
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Verification: Given f ,

- ▶ Check that $f(u) \in \{R, G, B\}$ for all $u \in V$, and
- ▶ Check each edge $\{u, v\}$ to make sure that $f(u) \neq f(v)$

NP

NP: decision problems where solutions can be *verified* in polynomial time.

Definition

A decision problem Q is in **NP** (*nondeterministic polynomial time*) if there exists a polynomial time algorithm $V(I, X)$ (called the *verifier*) such that

1. If I is a YES-instance of Q , then there is some X (usually called the *witness*, *proof*, or *solution*) with size polynomial in $|I|$ so that $V(I, X) = \text{YES}$.
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Examples:

- ▶ 3-coloring: Witness X is a coloring $f : V \rightarrow \{R, B, G\}$, verifier checks each edge $\{u, v\}$ to make sure $f(u) \neq f(v)$
 - ▶ If I is a YES instance, then there is a coloring so verifier will return YES
 - ▶ If I is a NO instance, then no valid coloring exists. Whatever X is, verifier returns NO.

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Examples:

- ▶ Max-Flow: Witness X is a flow $f : E \rightarrow \mathbb{R}_{\geq 0}$, verifier checks that it's feasible of value $\geq k$
 - ▶ If I is a YES instance, then there is a feasible flow of value at least k so verifier (on this flow) will return YES
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Examples:

- ▶ Factoring: Instance is pair of integers M, k . YES if M has as factor in $\{2, \dots, k\}$, NO otherwise.
 - ▶ Witness: integer f in $\{2, 3, \dots, k\}$. Verifier: returns YES if M/f is an integer and $f \in \{2, \dots, k\}$, NO otherwise.
 - ▶ If YES instance, then an f does exist so verifier returns YES on that f . If NO, then no such f exists so verifier always returns NO.

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Examples:

- ▶ Traveling Salesman: Instance is weighted graph G and integer k . YES iff G has a tour (walk that touches every vertex at least once) of length $\leq k$.
 - ▶ Witness: tour P . Verifier checks that it is a tour, has length at most k
 - ▶ If YES instance, then such a tour exists $\implies$ verifier returns YES on that tour.
 - ▶ If NO, no such tour exists $\implies$ verifier always returns NO.

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Important asymmetry: need a witness for YES, not a witness for NO.

P vs NP

Theorem

$$P \subseteq NP$$

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Proof.

Let $Q \in P$.

$V(I, X)$: Ignore X , solve on instance I .



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Question: Does $P = NP$, i.e., is $NP \subseteq P$?

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Proof.

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Question: Does $P = NP$, i.e., is $NP \subseteq P$?

- ▶ *Almost* everyone thinks no, but we don't know for sure!
- ▶ Not even particularly close to a proof.
- ▶ Think about what $P = NP$ would mean...



Reductions

Question: How could we prove that $P = NP$ or $P \neq NP$?

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- ▶ $P = NP$: Need to show that *every* problem in NP is also in P !
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Problem A is *polytime reducible* to problem B (written $A \leq_p B$) if, given a polynomial-time algorithm for B , we can use it to produce a polynomial-time algorithm for A .

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Means that B is “at least as hard” as A : if B is in P , then so is A .

- ▶ So “hardest” problems in NP are problems that many other problems reduce to.

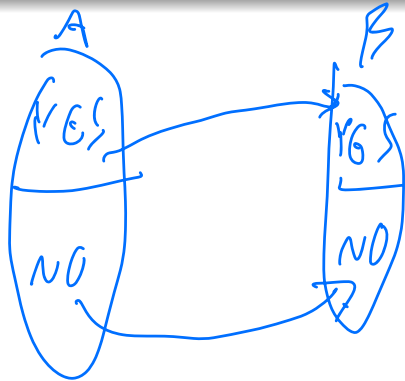
Many-One (Karp) Reductions

Almost always (and always in this course), use a special type of reduction.

Definition

A *Many-one* or *Karp* reduction from $\mathbf{A}$ to $\mathbf{B}$ is a function f which takes arbitrary instances of $\mathbf{A}$ and transforms them into instances of $\mathbf{B}$ so that

1. If x is a YES-instance of $\mathbf{A}$ then $f(x)$ is a YES-instance of $\mathbf{B}$.
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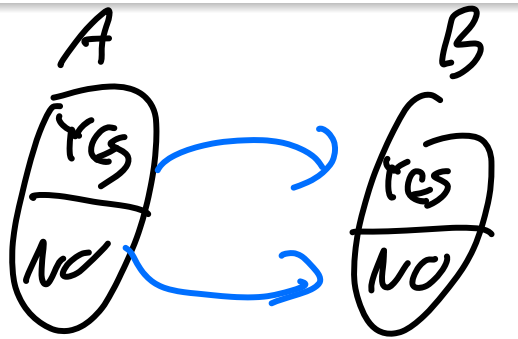
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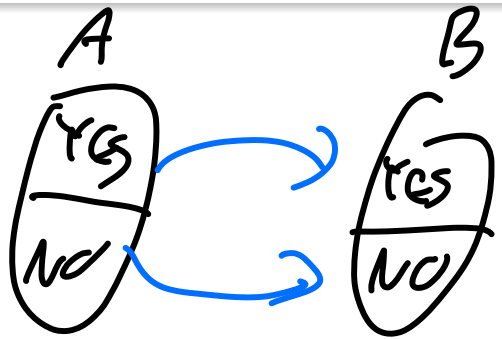
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So given instance x of $\mathbf{A}$, compute $f(x)$ and use polytime algorithm for $\mathbf{B}$ on $f(x)$

- ▶ Polytime, since f in polytime and algorithm for $\mathbf{B}$ in polytime
- ▶ Correct by first two properties of many-one reduction.

NP-Completeness

So what is “hardest problem” in **NP**?

Definition

Problem **Q** is **NP-hard** if $Q' \leq_p Q$ for all problems Q' in **NP**.

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Problem **Q** is **NP-complete** if it is **NP-hard** and in **NP**.

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So suppose **Q** is **NP-complete**.

- ▶ To prove $P \neq NP$: Hardest problem in **NP**! If anything in **NP** is not in **P**, then **Q** is not in **P**
- ▶ To prove $P = NP$: Just need to prove that $Q \in P$.

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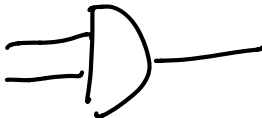
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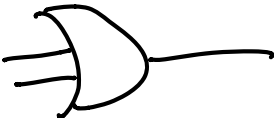
Is anything **NP-complete**?

Circuit-SAT

Definition

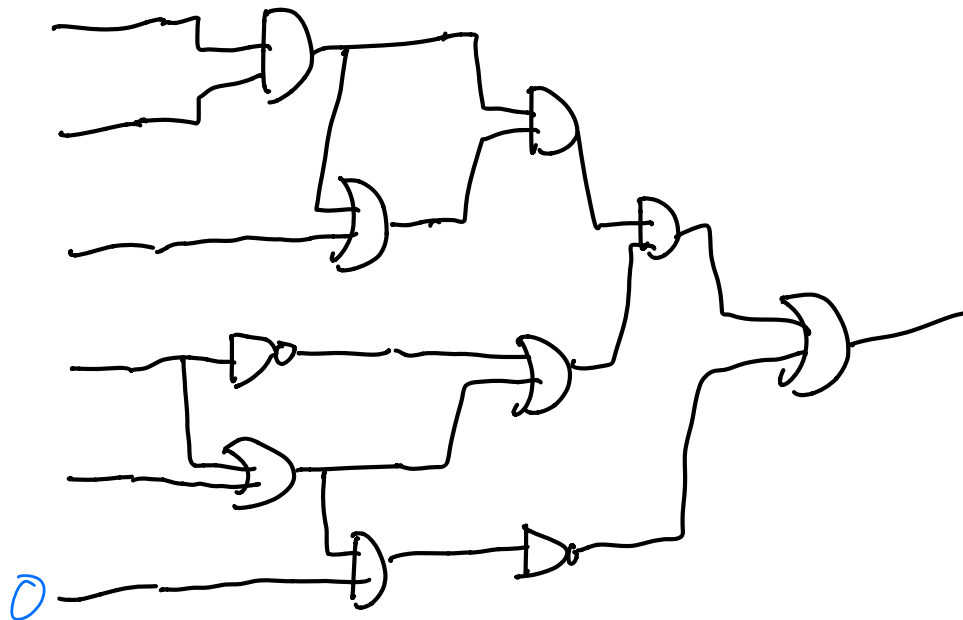
Circuit-SAT: Given a boolean circuit with a single output and no loops (some inputs might be hardwired), is there a way of setting the inputs so that the output of the circuit is **1**?

Gates: AND 

OR 

NOT 

Arbitrary fan-out



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Sketch of proof here. See book for details.

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Witness is a T/F (or 1/0) assignment to inputs. Verifier simulates circuit on assignment, checks that it outputs **1**.

- ▶ If input is a YES instance then there is some assignment so circuit outputs **1**. When verifier run on that assignment, returns YES.
- ▶ In input is a NO instance then in every assignment circuit outputs **0**. So verifier returns NO on every witness.



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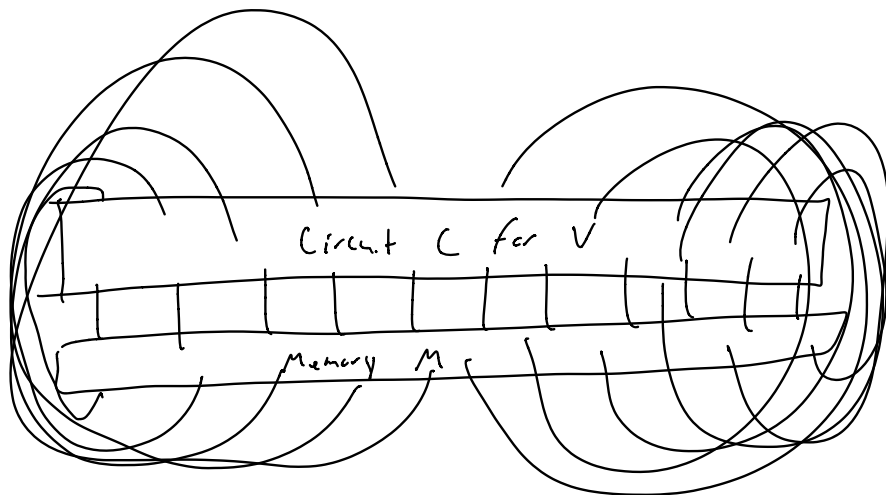
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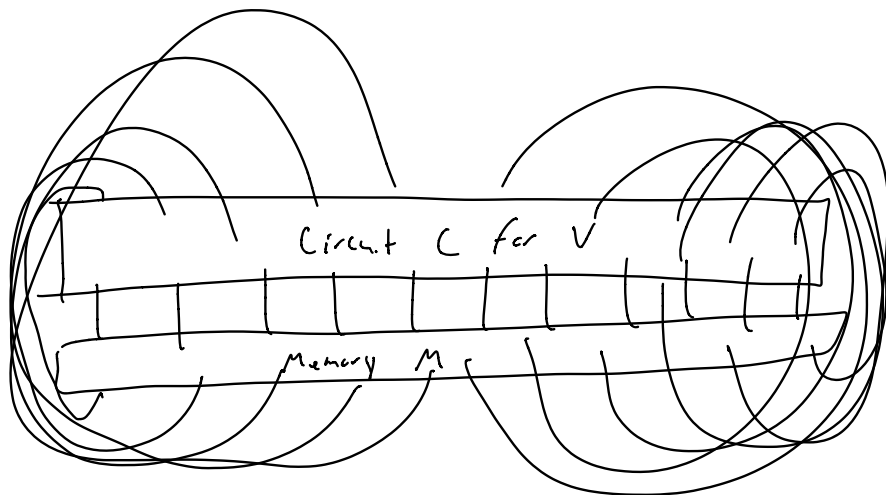
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Not a boolean circuit in Circuit-SAT sense: loops (feedback)

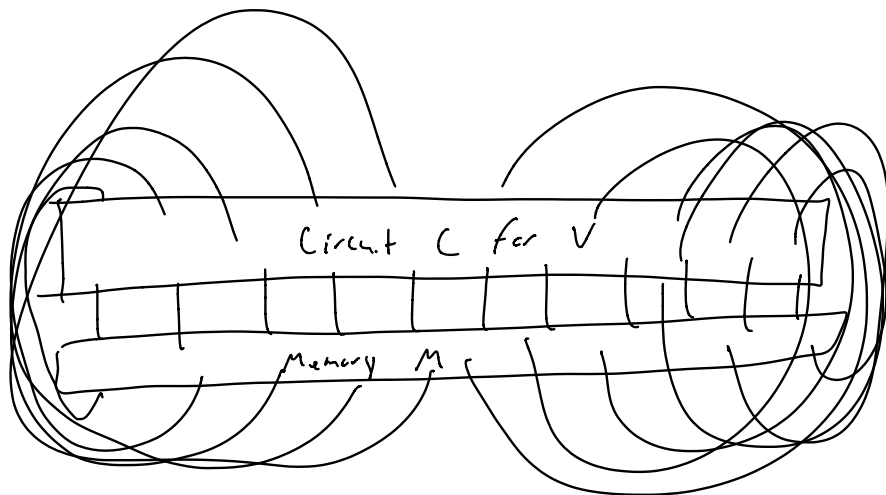
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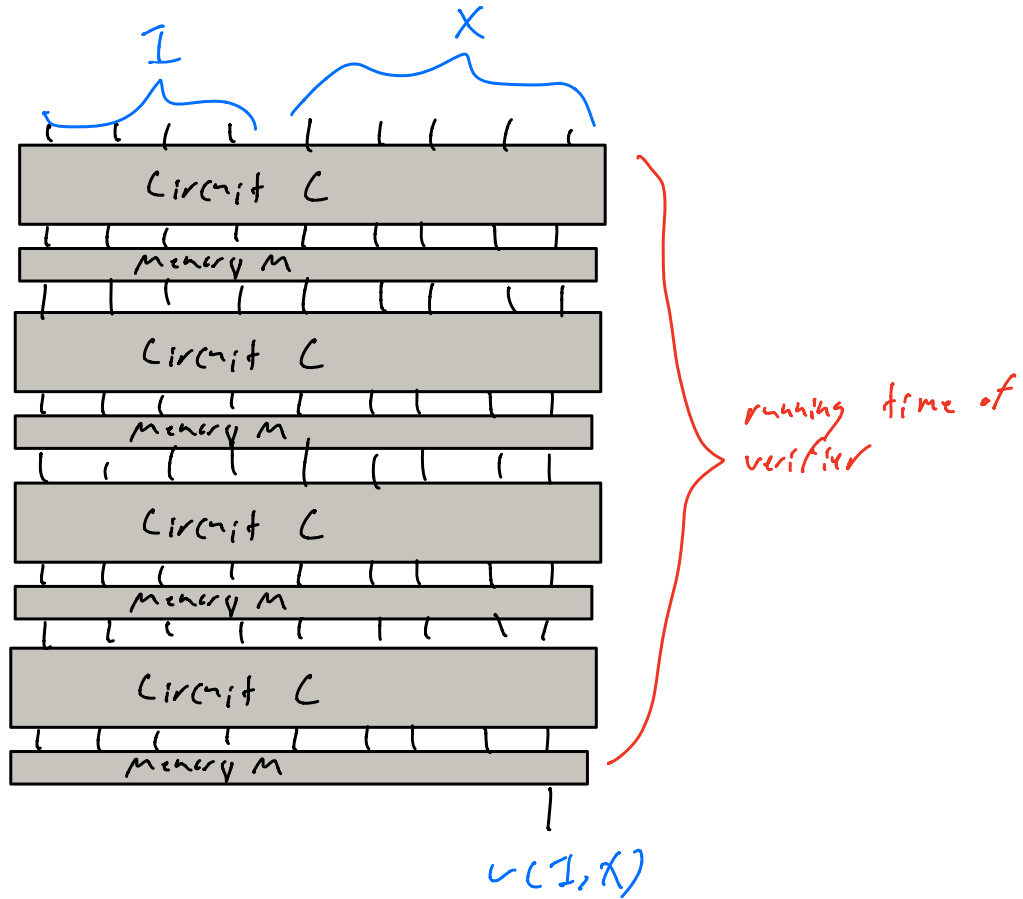
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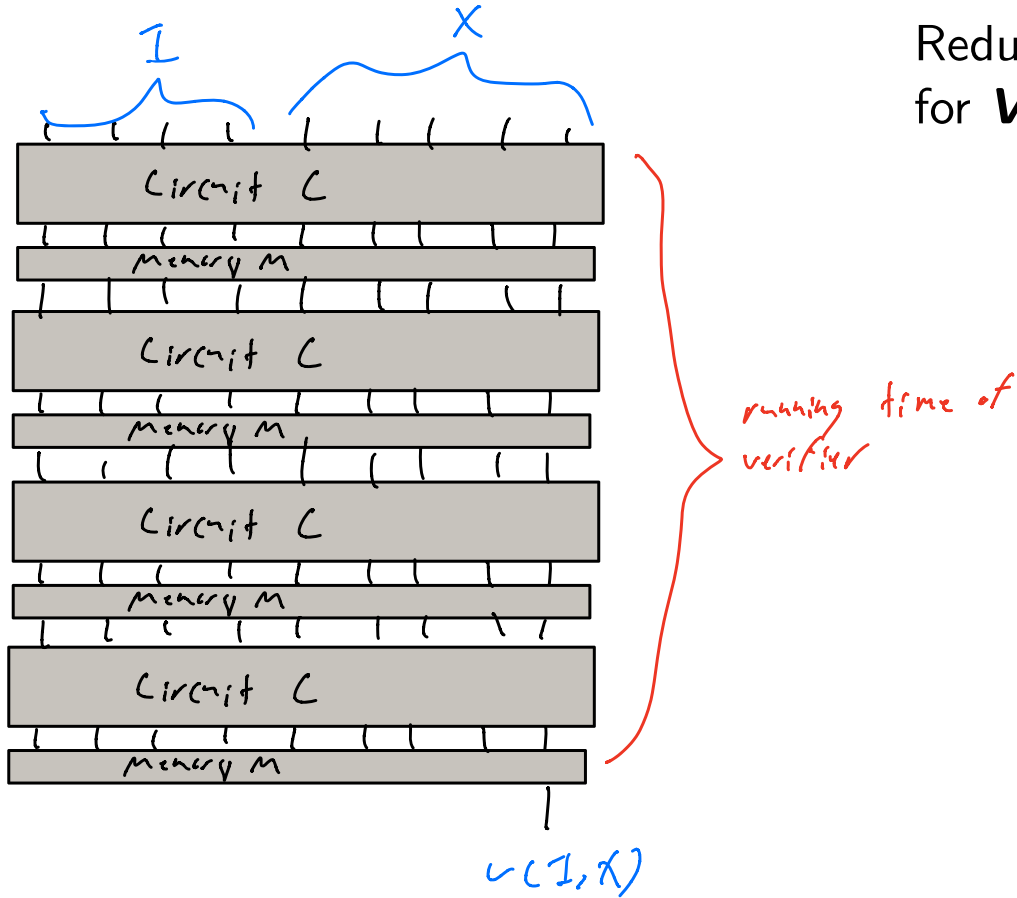
Fix: “Unroll” circuit using fact that $\mathbf{V}$ runs in polynomial time

Reduction

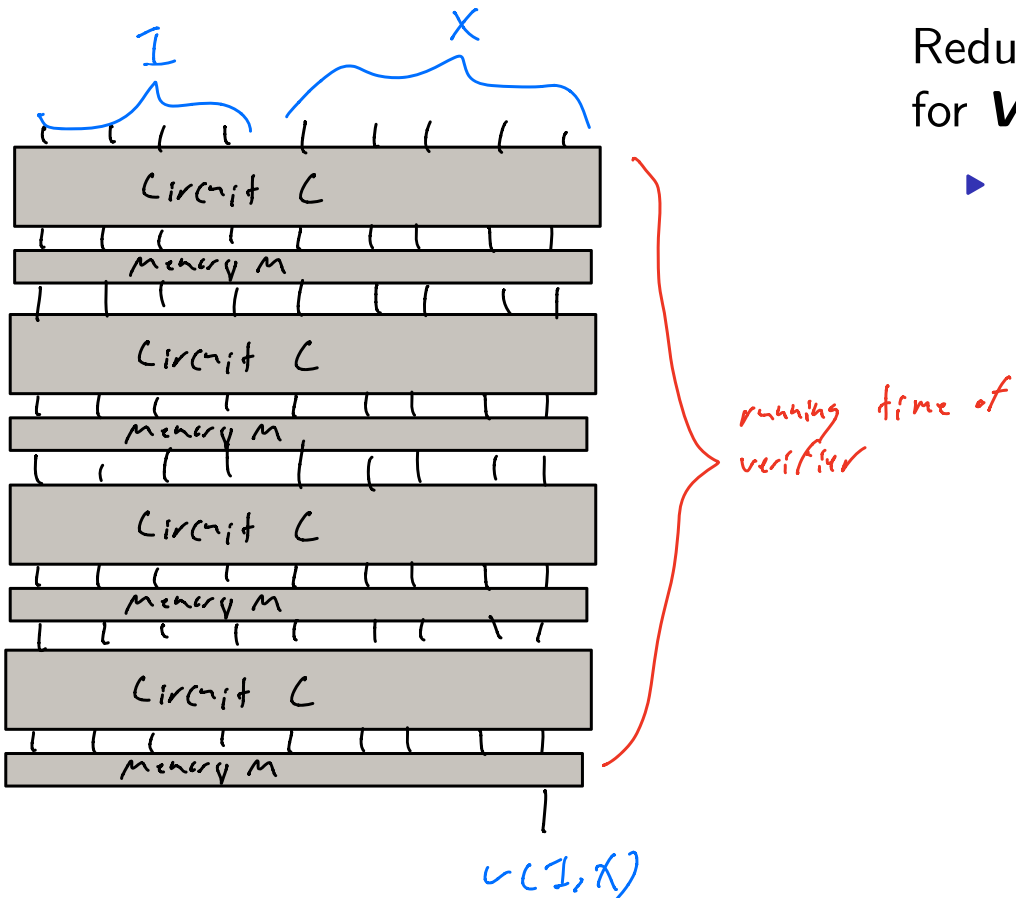


Reduction

Reduction: given instance I of A , construct this circuit for V , hardwire I . Combined circuit $f(I)$



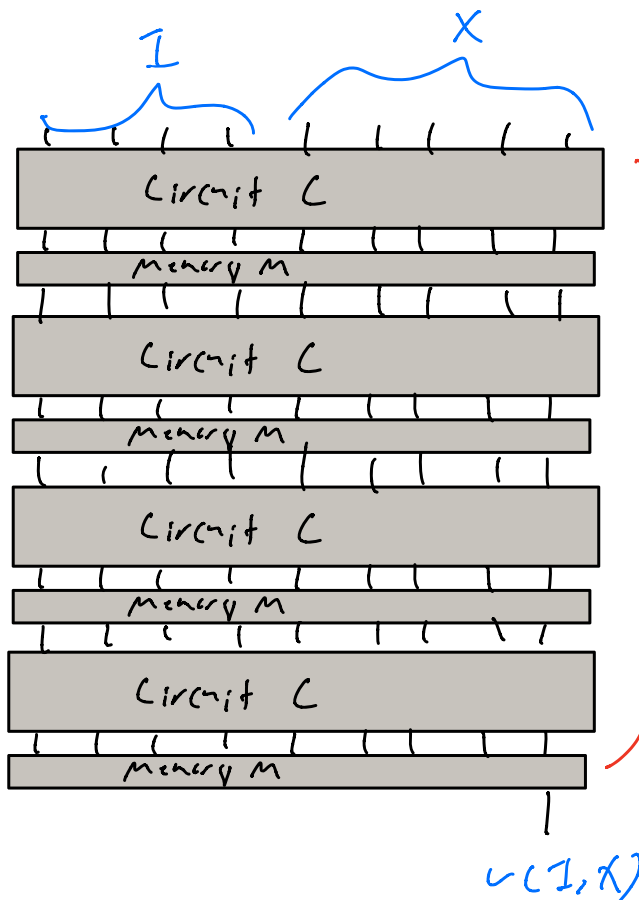
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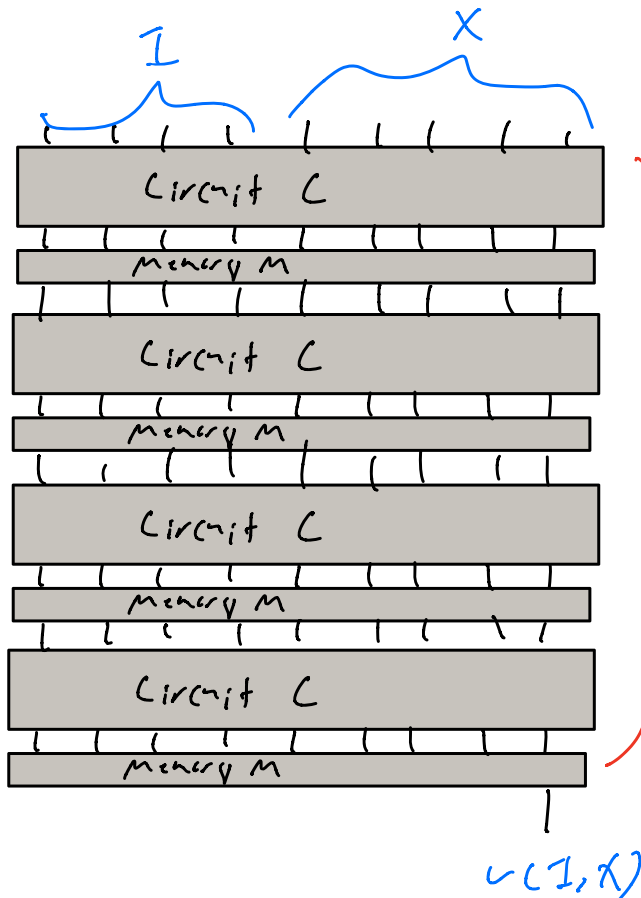
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- ▶ Polytime since V runs in polytime
- ▶ If I YES of A : there is some X so that $V(I, X) = \text{YES}$
 $\implies$ some X so that when X input to $f(I)$, outputs 1
 $\implies f(I)$ YES instance of Circuit-SAT.

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 $\implies$ some X so that when X input to $f(I)$, outputs **1**
 $\implies f(I)$ YES instance of Circuit-SAT.
- ▶ If I NO of A : For every X , know that $V(I, X) = \text{NO}$
 $\implies$ for every X , when X input to $f(I)$, outputs **0**
 $\implies f(I)$ NO instance of Circuit-SAT