

Lecture 25: Online Learning

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601.433/633 Introduction to Algorithms

Slides by Michael Dinitz and Jessica Sorrell

Introduction

Machine Learning from the point of view of theoretical computer science

- ▶ Proofs about performance
- ▶ Minimize assumptions
- ▶ *Not* going to talk about useful in practice, etc.

Today:

- ▶ Online Learning

Online Learning

Online Learning

Learning over time, not just one-shot

- ▶ See data one piece at a time
- ▶ Try to use historical data to make decisions as we go
- ▶ We don't assume data comes from a distribution. Could be adversarially chosen!

Example: Learning From Expert Advice

Intuition: stock market

- ▶ **N** experts
- ▶ Every day:
 - ▶ Every expert predicts up/down
 - ▶ Algorithm makes prediction
 - ▶ Find out what happened

What can/should we do? Can we always make an accurate prediction?

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- ▶ No! Experts could all be essentially random, uncorrelated with market

Easier (but still interesting) goal: can we do as well as the best expert?

- ▶ Don't try to learn the market: learn which expert knows the market best

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- ▶ Each mistake decreases # experts by **$1/2$**

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Weighted Majority

- ▶ Initialize all experts to weight **1**
- ▶ Predict based on *weighted* majority vote
- ▶ Penalize mistakes by cutting weights in half

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$$W \leq N(3/4)^M$$

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$$\implies (1/2)^m \leq W \leq N(3/4)^M \implies (4/3)^M \leq N2^m$$

$$\implies M \leq \log_{4/3}(N2^m) = \frac{m + \log N}{\log(4/3)} \approx 2.4(m + \log N)$$

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A Better Algorithm (and More General Framework!)

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General setup:

- ▶ T time steps (days)
- ▶ N actions the algorithm can take (experts)
- ▶ At each time step $t \in [T]$, algorithm A chooses an action $i \in [N]$
- ▶ Each action $i \in [N]$ then receives a loss $\ell_i^t \in [0, 1]$, and the algorithm receives loss $\ell_A^t = \ell_i^t$ corresponding to the action i that it chose at time t

Regret

Our new goal is to minimize regret.

Definition (Regret)

For all $t \in [T]$, let ℓ_A^t be the loss suffered by algorithm \mathbf{A} at time t . Then the *regret* of algorithm \mathbf{A} is

$$\text{Regret}(\mathbf{A}) = \frac{1}{T} \sum_{t=1}^T \ell_A^t - \frac{1}{T} \min_{i \in [N]} \sum_{t=1}^T \ell_i^t$$

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An algorithm is a *no-regret* algorithm if its regret goes to 0 as $T \rightarrow \infty$.

Multiplicative Weights Algorithm

Algorithm Multiplicative Weights (MW)

For $i \in [N]$, let $w_i^1 = 1$ be the weight of action i at time 1.

for $t = 1, \dots, T$ **do**

Let $W^t = \sum_{i \in [N]} w_i^t$ be the total weight at time t .

Choose action $i \in [N]$ at random according to the distribution $D(i) = \frac{w_i^t}{W^t}$

Pay loss ℓ_j^t for action i at time t

Update weights: $w_j^{t+1} \leftarrow w_j^t \cdot e^{-\epsilon \ell_j^t}$ for all $j \in [N]$

end for

Multiplicative Weights Analysis

Theorem

MW is a no-regret algorithm. Specifically, it has expected regret $O(\epsilon + \frac{\log(N)}{\epsilon T})$. That is,

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\ell_A^t] - \frac{1}{T} \min_{i \in [N]} \sum_{t=1}^T \ell_i^t \in O\left(\epsilon + \frac{\ln(N)}{\epsilon T}\right)$$

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If we set $\epsilon = \sqrt{\frac{\ln N}{T}}$, we get that the expected regret of MW is at most $2\sqrt{\frac{\ln N}{T}}$. This means that MW is a no-regret algorithm, since its regret $\rightarrow 0$ as $T \rightarrow \infty$.

Proof Sketch

Let $\mathbf{W}_t = \sum_{i \in [N]} \mathbf{w}_i^t$

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$$\mathbf{W}_{T+1} \leq N \exp(\epsilon^2 T - \epsilon \sum_{t=1}^T \mathbb{E}[\ell_A^t])$$

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$$\exp(-\epsilon \sum_{t=1}^T \ell_i^t) = \mathbf{w}_i^{T+1} \leq \mathbf{W}_{T+1} \leq N \exp(\epsilon^2 T - \epsilon \sum_{t \in [T]} \mathbb{E}[\ell_A^t])$$

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Proof Sketch

Still need to show that $\mathbf{W}_{T+1} \leq N \exp(\varepsilon^2 T - \varepsilon \sum_{t \in [T]} \mathbb{E}[\ell_A^t])$

$$\begin{aligned} \mathbf{W}_{t+1} &= \sum_{i \in [N]} \mathbf{w}_i^{t+1} = \sum_{i \in [N]} \mathbf{w}_i^t \exp(-\varepsilon \ell_i^t) &= (1 + \varepsilon^2) \mathbf{W}_t - \varepsilon \mathbf{W}_t \sum_{i \in [N]} \frac{\mathbf{w}_i^t}{\mathbf{W}_t} \ell_i^t && \text{by def. of } \mathbf{W}_t \\ &\leq \sum_{i \in [N]} \mathbf{w}_i^t \cdot (1 - \varepsilon \ell_i^t + \varepsilon^2 \ell_i^{t^2}) &= \mathbf{W}_t (1 + \varepsilon^2 - \varepsilon \sum_{i \in [N]} \frac{\mathbf{w}_i^t}{\mathbf{W}_t} \ell_i^t) \\ &\leq \sum_{i \in [N]} \mathbf{w}_i^t \cdot (1 - \varepsilon \ell_i^t + \varepsilon^2) &\leq \mathbf{W}_t \exp(\varepsilon^2 - \varepsilon \sum_{i \in [N]} \frac{\mathbf{w}_i^t}{\mathbf{W}_t} \ell_i^t) && \exp(x) \geq 1 + x \\ &= (1 + \varepsilon^2) \sum_{i \in [N]} \mathbf{w}_i^t - \varepsilon \sum_{i \in [N]} \mathbf{w}_i^t \ell_i^t &= \mathbf{W}_t \exp(\varepsilon^2 - \varepsilon \mathbb{E}[\ell_A^t]) && \text{by def. of } \mathbb{E}[\ell_A^t] \end{aligned}$$

Unrolling over all $t \in [T]$:

$$\begin{aligned} \Rightarrow \mathbf{W}_{T+1} &\leq \mathbf{W}_1 \cdot \prod_{t \in [T]} \exp(\varepsilon^2 - \varepsilon \mathbb{E}[\ell_A^t]) \\ &= N \exp(\varepsilon^2 T - \varepsilon \sum_{t \in [T]} \mathbb{E}[\ell_A^t]) \end{aligned}$$

Summary

- ▶ Online learning models learning problems where data arrives “online” (one at a time)
- ▶ Given a set of actions to choose from, we want to learn a good sequence of actions to take so that we do not incur too much loss
- ▶ We can analyze the performance of online learning algorithms using the notion of *regret*: how well did the algorithm perform compared to the best action in hindsight?
- ▶ We showed the multiplicative weights algorithm is a no-regret algorithm (its expected regret goes to **0** as $T \rightarrow \infty$).