Lecture 3: Intro to proofs for algorithms

Jessica Sorrell

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601.433/633 Introduction to Algorithms

Announcements

- Grading policy change for quizzes: drop two lowest scores
- First homework released today!
  - Due Monday, Sep 15, 11:59pm
- Course staff change: Nate Robinson no longer part of the course staff
- More office hours on course webpage / calendar, including Yan Zhong's recitation-like office hours (Wed 6-7pm, Malone 107)

Today

Discuss common proof techniques for algorithms.

- Inductive arguments (weak, strong)
- Proof by contradiction
- Direct proof
- Loop invariant
- Proof by contrapositive

We'll demonstrate proof techniques by proving the correctness and running time of algorithms you've seen before.

Quicksort review

$A = [4 \mid 5 \mid 3 \mid 2 \mid 8 \mid 0]$

↑ p

Algorithm Quicksort

Input: array $\boldsymbol{A}$ of length $\boldsymbol{n}$

if $\boldsymbol{n} \leq 1$ **then**
    return $\boldsymbol{A}$
end if
Pick some element $\boldsymbol{p} \in \boldsymbol{A}$ as the *pivot*
Let $\boldsymbol{L}$ be the elements less than or equal to $\boldsymbol{p}$, let $\boldsymbol{G}$ be the elements larger than $\boldsymbol{p}$
$\boldsymbol{L}' \leftarrow$ Quicksort($\boldsymbol{L}$)
$\boldsymbol{G}' \leftarrow$ Quicksort($\boldsymbol{G}$)
return $\boldsymbol{L}' \| \boldsymbol{p} \| \boldsymbol{G}'$

$L = [3, 2, 0]$

$G = [5, 8]$

$L' = [0, 2, 3]$ $\quad G' = [5, 8]$ $\quad [0, 2, 3, 4, 5, 8]$

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- e.g. Assume Quicksort always returns a sorted array for input arrays of size $\leq k$. Show it returns a sorted array for input arrays of size $k + 1$.

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- e.g. Quicksort always returns a sorted array for input arrays of size $\boldsymbol{n} \leq \mathbf{1}$
- Assume inductive hypothesis, that property holds *for all* $\boldsymbol{n} \leq \boldsymbol{k}$. Then show that property holds for $\boldsymbol{n} = \boldsymbol{k} + \mathbf{1}$.
- e.g. Assume Quicksort always returns a sorted array for input arrays of size $\leq \boldsymbol{k}$. Show it returns a sorted array for input arrays of size $\boldsymbol{k} + \mathbf{1}$.

Weak induction:

- Prove property holds for a base case
- Assume inductive hypothesis, that property holds for $\boldsymbol{n} = \boldsymbol{k}$. Then show that property holds for $\boldsymbol{n} = \boldsymbol{k} + \mathbf{1}$.
- e.g. Assume Quicksort always returns a sorted array for input arrays of size *exactly* $\boldsymbol{k}$. Show it returns a sorted array for input arrays of size $\boldsymbol{k} + \mathbf{1}$.

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  - L and G are of length $\leq n$, so by inductive hypothesis, Quicksort(L) and Quicksort(G) return sorted arrays L' and G'.

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  - Therefore $L' \| p \| G'$ is sorted.

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- A strong inductive hypothesis assumes the desired property holds for all $n \leq k$.
- Quicksort recursively calls itself on L and G, which we don't know the size of a priori
- In strong induction, we assume that Quicksort is correct for all arrays of size $\leq k$, so doesn't matter what the exact size L and G are, because we know they are both $\leq k$.

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- Note $n \geq 2$.

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- Note $n \geq 2$.
- So Quicksort(A) picks a pivot element $p \in A$, defines L and G as the elements less than or equal to p and the elements greater than p respectively, and recursively calls Quicksort on L and G .

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- Note $\boldsymbol{n} \geq \mathbf{2}$.
- So Quicksort($\boldsymbol{A}$) picks a pivot element $\boldsymbol{p} \in \boldsymbol{A}$, defines $\boldsymbol{L}$ and $\boldsymbol{G}$ as the elements less than or equal to $\boldsymbol{p}$ and the elements greater than $\boldsymbol{p}$ respectively, and recursively calls Quicksort on $\boldsymbol{L}$ and $\boldsymbol{G}$.
- By assumption that $\boldsymbol{A}$ is the smallest such array, $\boldsymbol{L}$ and $\boldsymbol{G}$ are sorted.

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- Note $n \geq 2$.
- So Quicksort(A) picks a pivot element $p \in A$, defines L and G as the elements less than or equal to p and the elements greater than p respectively, and recursively calls Quicksort on L and G .
- By assumption that A is the smallest such array, L and G are sorted.
- Therefore $L||p||G$ is sorted.
- Contradiction: A is not the smallest array such that Quicksort does not return a sorted array.

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For a statement of the form $\boldsymbol{A} \Rightarrow \boldsymbol{B}$, a direct proof shows that $\boldsymbol{B}$ follows from the logical implications of $\boldsymbol{A}$.

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- The worst case for runtime occurs when the pivot is the smallest or largest element of the array.
- In this case, the array is partitioned into an array of size $n-1$ and an array of size 0.
- This gives a recurrence $T(n) = T(n-1) + \Theta(n)$, which has solution $T(n) = \Theta(n^2)$.

Insertion Sort Review

4, 5, 8, 2, 1, 6

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Algorithm Insertion Sort
Input: array A of length n

```
for i ← 2 to n do
    j ← i
    while j > 1 and A[j] < A[j − 1] do
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- Initialization: the property is true at the start of the loop.
- Maintenance: if the property is true at the beginning of an iteration, it is true at beginning of the next iteration.
- Termination: when the loop terminates, the invariant holds and shows that the algorithm is correct.

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Claim: Given an array $\boldsymbol{A}$ of length $\boldsymbol{n}$, InsertionSort($\boldsymbol{A}$) returns an array with all elements of $\boldsymbol{A}$ sorted from least to greatest.

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Proof:

- Loop invariant: at iteration i, $A[1, i-1]$ contains all elements of the original input array $A[1, i-1]$, and is sorted.

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- Loop invariant: at iteration i, $A[1, i-1]$ contains all elements of the original input array $A[1, i-1]$, and is sorted.
- Initialization - At the beginning of the first iteration $i = 2$, $A[1]$ is sorted.
- Maintenance - In a single iteration, element $A[i]$ of the input Array is moved to the left until it is no longer smaller than the element to its left, therefore at the beginning of the next iteration, $A[1, i]$ is sorted and contains exactly the same elements as $A[1, i]$ from the original input array.

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- Initialization - At the beginning of the first iteration $i = 2$, $A[1]$ is sorted.
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- Termination - When the loop terminates, $i = n$ and therefore $A[1, n]$ is sorted and contains exactly the same elements as $A[1, n]$ from the original input array. Therefore the original input array has been sorted.

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Proof by contrapositive is a proof technique for conditional statements. That is, statements of the form "If $\boldsymbol{A}$, then $\boldsymbol{B}$."

It relies on the fact that $\boldsymbol{A} \Rightarrow \boldsymbol{B}$ is logically equivalent to $\neg \boldsymbol{B} \Rightarrow \neg \boldsymbol{A}$.

To prove $\boldsymbol{A} \Rightarrow \boldsymbol{B}$ by contrapositive, we show that if the negation of the conclusion is true ($\neg \boldsymbol{B}$), then the negation of the hypothesis is true ($\neg \boldsymbol{A}$).

Correctness of (one iteration of) Insertion Sort - Proof by Contrapositive

Claim: If the ith iteration of the inner WHILE loop terminates with counter value j for $j > 1$, then element $A[i]$ of the original input array is greater than or equal to $A[j-1]$.

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- A: The ith iteration of the inner WHILE loop terminates with counter value j for $j > 1$
- B: Element $A[i]$ of the original input array is greater than or equal to $A[j-1]$

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- Want to prove $A \Rightarrow B$

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- B: Element $A[i]$ of the original input array is greater than or equal to $A[j-1]$
- Want to prove $A \Rightarrow B$
- So will argue that if element $A[i]$ of the original input array is less than $A[j-1]$, then the ith iteration of the inner WHILE loop will not terminate with counter value j for $j > 1$.

Algorithm Insertion Sort
Input: array A of length n

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for i ← 2 to n do
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Claim: If element $\boldsymbol{A}[\boldsymbol{i}]$ of the original input array is less than $\boldsymbol{A}[\boldsymbol{j}-\mathbf{1}]$, then the $\boldsymbol{i}$th iteration of the inner WHILE loop will not terminate with counter value $\boldsymbol{j}$ for $\boldsymbol{j} > \mathbf{1}$.

- In order for the loop to terminate at counter value $\boldsymbol{j} > \mathbf{1}$, it must hold that $\boldsymbol{A}[\boldsymbol{j}] \geq \boldsymbol{A}[\boldsymbol{j}-\mathbf{1}]$.

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Claim: If element $A[i]$ of the original input array is less than $A[j-1]$, then the ith iteration of the inner WHILE loop will not terminate with counter value j for $j > 1$.

- In order for the loop to terminate at counter value $j > 1$, it must hold that $A[j] \geq A[j-1]$.
- Note that inside the WHILE loop, $A[j] = A[i]$ of the original input array. Therefore if $A[i] = A[j] < A[j-1]$, the loop will not terminate with counter value j.

Algorithm Insertion Sort

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  while $j > 1$ and $A[j] < A[j-1]$ **do**
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    $j \leftarrow j - 1$
  end while
end for