Lecture 3: Intro to proofs for algorithms

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September 2, 2025 601.433/633 Introduction to Algorithms

Announcements

- Grading policy change for quizzes: drop two lowest scores
- First homework released today!
 - Due Monday, Sep 15, 11:59pm
- Course staff change: Nate Robinson no longer part of the course staff
- ▶ More office hours on course webpage / calendar, including Yan Zhong's recitation-like office hours (Wed 6-7pm, Malone 107)

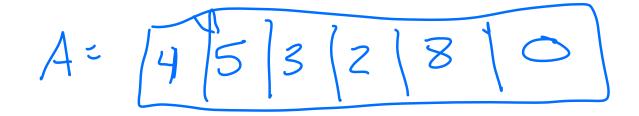
Today

Discuss common proof techniques for algorithms.

- Inductive arguments (weak, strong)
- Proof by contradiction
- Direct proof
- Loop invariant
- Proof by contrapositive

We'll demonstrate proof techniques by proving the correctness and running time of algorithms you've seen before.

Quicksort review



Algorithm Quicksort

Input: array \boldsymbol{A} of length \boldsymbol{n}

- 1: if $n \le 1$ then
- 2: return A
- 3: end if
- 4: Pick some element $p \in A$ as the pivot
- 5: Let ${m L}$ be the elements less than or equal to ${m p}$, let ${m G}$ be the elements larger than ${m p}$
- 6: $L' \leftarrow Quicksort(L)$
- 7: $G' \leftarrow Quicksort(G)$
- 8: return L'||p||G'

1 2 0,2,3

0, 2,3,4,5,8

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Weak induction:

- Prove property holds for a base case
- Assume inductive hypothesis, that property holds for n = k. Then show that property holds for n = k + 1.
- e.g. Assume Quicksort always returns a sorted array for input arrays of size exactly k. Show it returns a sorted array for input arrays of size k + 1.

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- ▶ A strong inductive hypothesis assumes the desired property holds for all $n \le k$.
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- In strong induction, we assume that Quicksort is correct for all arrays of size $\leq k$, so doesn't matter what the exact size L and G are, because we know they are both $\leq k$.

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Proof:

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- ightharpoonup By assumption that $m{A}$ is the smallest such array, $m{L}$ and $m{G}$ are sorted.
- ▶ Therefore L||p||G is sorted.
- Contradiction: A is not the smallest array such that Quicksort does not return a sorted array.

Direct Proof

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For a statement of the form $A \Rightarrow B$, a direct proof shows that B follows from the logical implications of A.

Claim: Quicksort runs in time $O(n^2)$ in the worst case.

• Before making its two recursive calls, Quicksort compares every element of its input array to the pivot, taking time $\Theta(n)$.

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- ▶ The worst case for runtime occurs when the pivot is the smallest or largest element of the array.
- ▶ In this case, the array is partitioned into an array of size n-1 and an array of size 0.
- ▶ This gives a recurrence $T(n) = T(n-1) + \Theta(n)$, which has solution $T(n) = \Theta(n^2)$.

Insertion Sort Review

Algorithm Insertion Sort Input: array **A** of length **n**

1: for
$$i \leftarrow 2$$
 to n do

3: while
$$j>1$$
 and $oldsymbol{A}[j] do$

4: Swap
$$A[j]$$
 and $A[j-1]$

5:
$$j \leftarrow j - 1$$

4,2,5,8,

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- Initialization: the property is true at the start of the loop.
- ▶ Maintenance: if the property is true at the beginning of an iteration, it is true at beginning of the next iteration.
- ▶ Termination: when the loop terminates, the invariant holds and shows that the algorithm is correct.

Correctness of Insertion Sort - Proof by Loop Invariant

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Loop invariant: at iteration i, A[1, i-1] contains all elements of the original input array A[1, i-1], and is sorted.

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- Loop invariant: at iteration i, A[1, i-1] contains all elements of the original input array A[1, i-1], and is sorted.
- ▶ Initialization At the beginning of the first iteration i = 2, A[1] is sorted.
- Maintenance In a single iteration, element A[i] of the input Array is moved to the left until it is no longer smaller than the element to its left, therefore at the beginning of the next iteration, A[1,i] is sorted and contains exactly the same elements as A[1,i] from the original input array.

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- ▶ Termination When the loop terminates, i = n and therefore A[1, n] is sorted and contains exactly the same elements as A[1, n] from the original input array. Therefore the original input array has been sorted.

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It relies on the fact that $A \Rightarrow B$ is logically equivalent to $\neg B \Rightarrow \neg A$.

To prove $A \Rightarrow B$ by contrapositive, we show that if the negation of the conclusion is true $(\neg B)$, then the negation of the hypothesis is true $(\neg A)$.

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- ▶ Want to prove $A \Rightarrow B$
- So will argue that if element A[i] of the original input array is less than A[j-1], then the ith iteration of the inner WHILE loop will not terminate with counter value j for j > 1.

Algorithm Insertion Sort Input: array **A** of length **n**

- 1: for $i \leftarrow 2$ to n do
- 2: $j \leftarrow i$
- 3: **while** j > 1 and A[j] < A[j-1] **do**
- 4: Swap A[j] and A[j-1]
- 5: $j \leftarrow j 1$
- 6: end while
- 7: end for

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- In order for the loop to terminate at counter value j > 1, it must hold that $A[j] \ge A[j-1]$.
- Note that inside the WHILE loop, A[j] = A[i] of the original input array. Therefore if A[i] = A[j] < A[j-1], the loop will not terminate with counter value j.

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