Lecture 4: Probabilistic Analysis, Randomized Quicksort

Michael Dinitz

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601.433/633 Introduction to Algorithms

Introduction: Sorting

- Sorting: given array of comparable elements, put them in sorted order
- Popular topic to cover in Algorithms courses
- This course:
  - I assume you know the basics (mergesort, quicksort, insertion sort, selection sort, bubble sort, etc.) from Data Structures
  - Today: more advanced sorting (randomized quicksort)
  - Next week: Sorting lower bound and ways around it.

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Today: adding randomness into quicksort.

Quicksort Basics (Review)

Input: array A of length n.

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Input: array $\boldsymbol{A}$ of length $\boldsymbol{n}$.

Algorithm:

1. If $\boldsymbol{n} = \mathbf{0}$ or $\mathbf{1}$, return $\boldsymbol{A}$ (already sorted)
2. Pick some element $\boldsymbol{p}$ as the *pivot*
3. Compare every element of $\boldsymbol{A}$ to $\boldsymbol{p}$. Let $\boldsymbol{L}$ be the elements less than $\boldsymbol{p}$, let $\boldsymbol{G}$ be the elements larger than $\boldsymbol{p}$. Create array $[\boldsymbol{L}, \boldsymbol{p}, \boldsymbol{G}]$
4. Recursively sort $\boldsymbol{L}$ and $\boldsymbol{G}$.

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Not fully specified: how to choose $\boldsymbol{p}$?

- Traditionally: some simple deterministic choice (first element, last element, etc.)
- Next lecture: better deterministic choice (not very practical)
- Now: first element

Quicksort Analysis

Upper bound:

If $\boldsymbol{p}$ picked as pivot in step 2, then in correct place after step **3**

Quicksort Analysis

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If p picked as pivot in step 2, then in correct place after step 3

$\implies$ step 2 and 3 executed at most n times.

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Step $\mathbf{3}$ takes time $\boldsymbol{O(n)}$ (compare every element to pivot)

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$\implies$ $\boldsymbol{p} = \boldsymbol{A}[\mathbf{0}]$ is smallest element

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$\implies$ running time is $T(n) = T(n-1) + cn \implies T(n) = \Theta(n^2)$

Randomized Quicksort

Randomized Quicksort: pick $\boldsymbol{p}$ *uniformly at random* from $\boldsymbol{A}$.

Today: prove that *expected* running time at most $\boldsymbol{O(n \log n)}$ for *every* input $\boldsymbol{A}$.

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Before doing analysis, quick review of basic probability theory.

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Only semi-formal here. Look at CLRS Chapter 5 and Appendix C, take Introduction to Probability

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- "Event that first die is 3": $\{(3, x) : x \in \{1, 2, \ldots, 6\}\}$
- "Event that dice add up to 7 or 11": $\{(x, y) \in \Omega : (x + y = 7) \text{ or } (x + y = 11)\}$

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Random Variable: $X : \Omega \to \mathbb{R}$

- X_1: value of first die. $X_1(x, y) = x$
- X_2: value of second die. $X_2(x, y) = y$
- $X = X_1 + X_2$: sum of the dice. $X(x, y) = x + y = X_1(x, y) + X_2(x, y)$

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Random variables super important! Running time of randomized quicksort is a random variable.

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Conditional probability: if A and B are events:

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]} = \frac{\sum_{e \in A \cap B} Pr[e]}{\sum_{e \in A} Pr[e]}$$

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Can be useful to rearrange terms to get different equation:

$$E[X] = \sum_{e \in \Omega} X(e) Pr[e] = \sum_{y \in \mathbb{R}} \sum_{e \in \Omega : X(e) = y} y \cdot Pr[e] = \sum_{y \in \mathbb{R}} y \cdot Pr[X = y]$$

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Conditional Expectation: A an event, X a random variable.

$$E[X|A] = \frac{1}{Pr[A]} \sum_{e \in A} X(e) Pr[e]$$

Linearity of Expectations

Amazing feature of expectations: linearity!

Theorem

For any two random variables $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, and any constants* α *and* β*:*
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Consider rolling two dice. Let X be sum. What is $E[X]$?

- $E[X] = \sum_{e \in \Omega} X(e) Pr[e]$. 36 term sum!
- $E[X] = \sum_{y \in \mathbb{R}} y \cdot Pr[X = y]$. What is $Pr[X = 2]$, $Pr[X = 3]$, …?

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$\implies \boldsymbol{E}[\boldsymbol{X}] = 3.5 + 3.5 = 7$

Linearity of Expectations: Proof

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Proof.

$$\boldsymbol{E}[\alpha\boldsymbol{X} + \beta\boldsymbol{Y}] = \sum_{e \in \Omega} \boldsymbol{Pr}[e]\,(\alpha\boldsymbol{X}(e) + \beta\boldsymbol{Y}(e))$$

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Holds no matter how correlated $\boldsymbol{X}$ and $\boldsymbol{Y}$ are!

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Definitions:

- $X = \#$ of comparisons (random variable)
- $e_i = i$'th smallest element (for $i \in \{1, \ldots, n\}$)
- X_{ij} random variable for all $i, j \in \{1, \ldots, n\}$ with $i < j$:

$$X_{ij} = \begin{cases} 1 & \text{if algorithm compares } e_i \text{ and } e_j \text{ at any point in time} \\ 0 & \text{otherwise} \end{cases}$$

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Algorithm never compares the same two elements more than once $\implies X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$

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- $i = 1, j = n$:

Randomized Quicksort II

Algorithm never compares the same two elements more than once $\implies X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$

$$E[X] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

So just need to understand $E[X_{ij}]$

Simple cases:

- $j = i + 1$: $X_{ij} = 1$ no matter what, so $E[X_{ij}] = 1$
- $i = 1, j = n$: e_1 and e_n compared if and only if first pivot chosen is e_1 or e_n $\implies E[X_{1n}] = \frac{2}{n}$

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So X_{ij} not determined until $e_i \le p \le e_j$, and when it is determined has $E[X_{ij}] = \frac{2}{j-i+1}$
$\implies E[X_{ij}] = \frac{2}{j-i+1}$

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$$\begin{aligned} E[X_{ij}] &= \sum_{k=1}^{n} E[X_{ij}|Y_k] Pr[Y_k] && (Y_k \text{ disjoint and partition } \Omega) \\ &= \frac{2}{j-i+1} \sum_{k=1}^{n} Pr[Y_k] \\ &= \frac{2}{j-i+1} \end{aligned}$$

Randomized Quicksort: Final Analysis

Expected running time of randomized quicksort:

$$\begin{aligned}
E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] && \text{(linearity of expectations)} \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= 2 \sum_{i=1}^{n-1} \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-i+1} \right) \\
&\leq 2 \sum_{i=1}^{n-1} H_n && \left(H_n = \sum_{j=1}^{n} \frac{1}{j} \right) \\
&\leq 2nH_n \\
&\leq O(n \log n)
\end{aligned}$$