Lecture 4: Probabilistic Analysis, Randomized Quicksort

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601.433/633 Introduction to Algorithms
Slides by Michael Dinitz

Introduction: Sorting

- Sorting: given array of comparable elements, put them in sorted order
- Popular topic to cover in Algorithms courses
- This course:
  - I assume you know the basics (mergesort, quicksort, insertion sort, selection sort, bubble sort, etc.) from Data Structures
  - Today: more advanced sorting (randomized quicksort)
  - Next week: Sorting lower bound and ways around it.

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- Still assume worst-case inputs, give bound on worst-case *expected* running time.

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Today: adding randomness into quicksort.

Quicksort Basics (Review)

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Algorithm:

1. If $\boldsymbol{n} = \mathbf{0}$ or $\mathbf{1}$, return $\boldsymbol{A}$ (already sorted)
2. Pick some element $\boldsymbol{p}$ as the *pivot*
3. Compare every element of $\boldsymbol{A}$ to $\boldsymbol{p}$. Let $\boldsymbol{L}$ be the elements less than $\boldsymbol{p}$, let $\boldsymbol{G}$ be the elements larger than $\boldsymbol{p}$.
4. Recursively sort $\boldsymbol{L}$ and $\boldsymbol{G}$. Return $\boldsymbol{L} \| \boldsymbol{p} \| \boldsymbol{G}$

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Not fully specified: how to choose $\boldsymbol{p}$?

- Traditionally: some simple deterministic choice (first element, last element, etc.)
- Next lecture: better deterministic choice (not very practical)
- Now: first element

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Lower Bound:
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$\implies$ $\boldsymbol{p} = \boldsymbol{A}[\mathbf{0}]$ is smallest element

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$\Longrightarrow$ $\boldsymbol{p} = \boldsymbol{A}[\mathbf{0}]$ is smallest element $\Longrightarrow$ $\boldsymbol{L} = \varnothing$ and $\boldsymbol{G} = \boldsymbol{A}[\mathbf{1..}\boldsymbol{n} - \mathbf{1}]$
$\Longrightarrow$ in one call to quicksort, do $\boldsymbol{\Omega(n)}$ work to compare everything to $\boldsymbol{p}$, then recurse on array of size $\boldsymbol{n} - \mathbf{1}$

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$\implies$ running time is $T(n) = T(n-1) + cn$

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$\implies$ in one call to quicksort, do $\Omega(n)$ work to compare everything to p, then recurse on array of size $n - 1$

$\implies$ running time is $T(n) = T(n-1) + cn \implies T(n) = \Theta(n^2)$

Randomized Quicksort

Randomized Quicksort: pick $\boldsymbol{p}$ *uniformly at random* from $\boldsymbol{A}$.

Today: prove that *expected* running time at most $\boldsymbol{O(n \log n)}$ for *every* input $\boldsymbol{A}$.

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Before doing analysis, quick review of basic probability theory.

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Only semi-formal here. Look at CLRS Chapter 5 and Appendix C, take Introduction to Probability

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- "Event that first die is $\mathbf{3}$": $\{(3, x) : x \in \{1, 2, \ldots, 6\}\}$
- "Event that dice add up to $\mathbf{7}$ or $\mathbf{11}$": $\{(x, y) \in \Omega : (x + y = 7) \text{ or } (x + y = 11)\}$

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- "Event that first die is 3": $\{(3, x) : x \in \{1, 2, \ldots, 6\}\}$
- "Event that dice add up to 7 or 11": $\{(x, y) \in \Omega : (x + y = 7) \text{ or } (x + y = 11)\}$

Random Variable: $X : \Omega \to \mathbb{R}$

- X_1: value of first die. $X_1(x, y) = x$
- X_2: value of second die. $X_2(x, y) = y$
- $X = X_1 + X_2$: sum of the dice. $X(x, y) = x + y = X_1(x, y) + X_2(x, y)$

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Random variables super important! Running time of randomized quicksort is a random variable.

Probability Basics II

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For each $\boldsymbol{e} \in \boldsymbol{\Omega}$ let $\boldsymbol{Pr}[\boldsymbol{e}]$ be probability of $\boldsymbol{e}$ (probability distribution)

- $\boldsymbol{Pr}[\boldsymbol{e}] \geq \mathbf{0}$ for all $\boldsymbol{e} \in \boldsymbol{\Omega}$, and $\sum_{\boldsymbol{e} \in \boldsymbol{\Omega}} \boldsymbol{Pr}[\boldsymbol{e}] = \mathbf{1}$
- Probability of an event $\boldsymbol{A}$ is $\boldsymbol{Pr}[\boldsymbol{A}] = \sum_{\boldsymbol{e} \in \boldsymbol{A}} \boldsymbol{Pr}[\boldsymbol{e}]$

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For each $e \in \Omega$ let $Pr[e]$ be probability of e (probability distribution)

- $Pr[e] \geq 0$ for all $e \in \Omega$, and $\sum_{e \in \Omega} Pr[e] = 1$
- Probability of an event A is $Pr[A] = \sum_{e \in A} Pr[e]$

Conditional probability: if A and B are events:

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]} = \frac{\sum_{e \in A \cap B} Pr[e]}{\sum_{e \in A} Pr[e]}$$

Probability Basics III: Expectations

Expectation of a random variable:

$$\boldsymbol{E}[\boldsymbol{X}] = \sum_{\boldsymbol{e} \in \boldsymbol{\Omega}} \boldsymbol{X}(\boldsymbol{e})\boldsymbol{Pr}[\boldsymbol{e}]$$

"Average" of the random variable according to probability distribution

Probability Basics III: Expectations

Expectation of a random variable:

$$E[X] = \sum_{e \in \Omega} X(e) Pr[e]$$

"Average" of the random variable according to probability distribution

Can be useful to rearrange terms to get different equation:

$$E[X] = \sum_{e \in \Omega} X(e) Pr[e] = \sum_{y \in \mathbb{R}} \sum_{e \in \Omega : X(e) = y} y \cdot Pr[e] = \sum_{y \in \mathbb{R}} y \cdot Pr[X = y]$$

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Conditional Expectation: $\boldsymbol{A}$ an event, $\boldsymbol{X}$ a random variable.

$$\boldsymbol{E[X|A] = \frac{1}{Pr[A]} \sum_{e \in A} X(e) Pr[e]}$$

Linearity of Expectations

Amazing feature of expectations: linearity!

Theorem

For any two random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$, and any constants α and β:
$\boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] = \alpha \boldsymbol{E}[\boldsymbol{X}] + \beta \boldsymbol{E}[\boldsymbol{Y}]$

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Consider rolling two dice. Let $\boldsymbol{X}$ be sum. What is $\boldsymbol{E}[\boldsymbol{X}]$?

- $\boldsymbol{E}[\boldsymbol{X}] = \sum_{e \in \Omega} \boldsymbol{X}(e) \boldsymbol{Pr}[e]$. 36 term sum!
- $\boldsymbol{E}[\boldsymbol{X}] = \sum_{y \in \mathbb{R}} y \cdot \boldsymbol{Pr}[\boldsymbol{X} = y]$. What is $\boldsymbol{Pr}[\boldsymbol{X} = 2]$, $\boldsymbol{Pr}[\boldsymbol{X} = 3]$, ...?

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Instead: $\boldsymbol{X} = \boldsymbol{X}_1 + \boldsymbol{X}_2$. So $\boldsymbol{E}[\boldsymbol{X}] = \boldsymbol{E}[\boldsymbol{X}_1 + \boldsymbol{X}_2] = \boldsymbol{E}[\boldsymbol{X}_1] + \boldsymbol{E}[\boldsymbol{X}_2]$

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Instead: $X = X_1 + X_2$. So $E[X] = E[X_1 + X_2] = E[X_1] + E[X_2]$

$$E[X_1] = E[X_2] = \sum_{y=1}^{6} \frac{1}{6} y = \frac{21}{6} = 3.5$$

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$\implies \boldsymbol{E}[\boldsymbol{X}] = 3.5 + 3.5 = 7$

Linearity of Expectations: Proof

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$\boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] = \alpha \boldsymbol{E}[\boldsymbol{X}] + \beta \boldsymbol{E}[\boldsymbol{Y}]$

Proof.

$$\boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] = \sum_{e \in \Omega} \boldsymbol{Pr}[e] \left(\alpha \boldsymbol{X}(e) + \beta \boldsymbol{Y}(e) \right)$$

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\end{aligned}$$

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&= \alpha \boldsymbol{E}[\boldsymbol{X}] + \beta \boldsymbol{E}[\boldsymbol{Y}]
\end{aligned}$$

Holds no matter how correlated $\boldsymbol{X}$ and $\boldsymbol{Y}$ are!

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Definitions:

- $X = \#$ of comparisons (random variable)
- $e_i = i$'th smallest element (for $i \in \{1, \ldots, n\}$)
- X_{ij} random variable for all $i, j \in \{1, \ldots, n\}$ with $i < j$:

$$X_{ij} = \begin{cases} 1 & \text{if algorithm compares } e_i \text{ and } e_j \text{ at any point in time} \\ 0 & \text{otherwise} \end{cases}$$

Randomized Quicksort II

Algorithm never compares the same two elements more than once $\implies \boldsymbol{X} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{X_{ij}}$

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- $j = i + 1$:

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So just need to understand $E[X_{ij}]$

Simple cases:

- $j = i + 1$: $X_{ij} = 1$ no matter what, so $E[X_{ij}] = 1$

Randomized Quicksort II

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$$E[X] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

So just need to understand $E[X_{ij}]$

Simple cases:

- $j = i + 1$: $X_{ij} = 1$ no matter what, so $E[X_{ij}] = 1$
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Simple cases:

- $j = i + 1$: $X_{ij} = 1$ no matter what, so $E[X_{ij}] = 1$
- $i = 1, j = n$: e_1 and e_n compared if and only if first pivot chosen is e_1 or e_n $\implies E[X_{1n}] = \frac{2}{n}$

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So X_{ij} not determined until $e_i \le p \le e_j$, and when it is determined has $E[X_{ij}] = \frac{2}{j-i+1}$
$\implies E[X_{ij}] = \frac{2}{j-i+1}$

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$$\begin{aligned} E[X_{ij}] &= \sum_{k=1}^{n} E[X_{ij}|Y_k] Pr[Y_k] && (Y_k \text{ disjoint and partition } \Omega) \\ &= \frac{2}{j-i+1} \sum_{k=1}^{n} Pr[Y_k] \\ &= \frac{2}{j-i+1} \end{aligned}$$

Randomized Quicksort: Final Analysis

Expected running time of randomized quicksort:

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] && \text{(linearity of expectations)} \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= 2\sum_{i=1}^{n-1} \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-i+1} \right) \\
&\le 2\sum_{i=1}^{n-1} H_n && \left(H_n = \sum_{j=1}^{n} \frac{1}{j} \right) \\
&\le 2nH_n \\
&\le O(n \log n)
\end{aligned}
$$