Lecture 6: Sorting Lower Bound and "Linear-Time" Sorting

Michael Dinitz

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601.433/633 Introduction to Algorithms

Introduction

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No: every algorithm in the comparison model must have worst-case running time $\Omega(n \log n)$.

Yes: If we assume extra structure for the elements, can do sorting in $O(n)$ time*

Sorting Lower Bound

Statement

Theorem

Any sorting algorithm in the comparison model must make at least $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

Lower bound on the number of comparisons – running time could be even worse!
Allows algorithm to reorder elements, copy them, move them, etc. for free.

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Why is this hard?

- Lower bound needs to hold for *all* algorithms
- How can we simultaneously reason about algorithms as different as mergesort, quicksort, heapsort, . . . ?

Sorting as Permutations

Think of an array $\boldsymbol{A}$ as a *permutation*: $\boldsymbol{A}[\boldsymbol{i}]$ is the $\boldsymbol{\pi}(\boldsymbol{i})$'th smallest element

$$\boldsymbol{A} = [\mathbf{23}, \mathbf{14}, \mathbf{2}, \mathbf{5}, \mathbf{76}]$$

Corresponds to $\boldsymbol{\pi} = (\mathbf{3}, \mathbf{2}, \mathbf{0}, \mathbf{1}, \mathbf{4})$:

$$\boldsymbol{\pi}(\mathbf{0}) = \mathbf{3} \qquad \boldsymbol{\pi}(\mathbf{1}) = \mathbf{2} \qquad \boldsymbol{\pi}(\mathbf{2}) = \mathbf{0} \qquad \boldsymbol{\pi}(\mathbf{3}) = \mathbf{1} \qquad \boldsymbol{\pi}(\mathbf{4}) = \mathbf{4}$$

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Lemma

Given A with $|A| = n$, if can sort in $T(n)$ comparisons then can find π in $T(n)$ comparisons

Sorting As Permutations (cont'd)

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Proof Sketch.

- "Tag" each element of A with index:
  $[23, 14, 2, 5, 76] \rightarrow [(23, 0), (14, 1), (2, 2), (5, 3), (76, 4)]$
- Sort tagged A into tagged B with $T(n)$ comparisons:
  $[(2, 2), (5, 3), (14, 1), (23, 0), (76, 4)]$
- Iterate through to get π: $\pi(2) = 0, \pi(3) = 1, \pi(1) = 2, \pi(0) = 3, \pi(4) = 4$ $\square$

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Corollary

If need at least $T(n)$ comparisons to find π, need at least $T(n)$ comparisons to sort!

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Want to show that it takes $\Omega(n \log n)$ comparisons to find π in comparison model.

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Arbitrary algorithm:

- Starts with some comparison (e.g., compares $A[0]$ to $A[1]$)
- Rules out some possible permutations!
  - If $A[0] < A[1]$ then $\pi(0) < \pi(1)$
  - If $A[0] > A[1]$ then $\pi(0) > \pi(1)$
- Depending on outcome, choose next comparison to make.
- Continue until only one possible permutation.

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Remind you of anything?

Decision Trees

Model any algorithm as a *binary decision tree*

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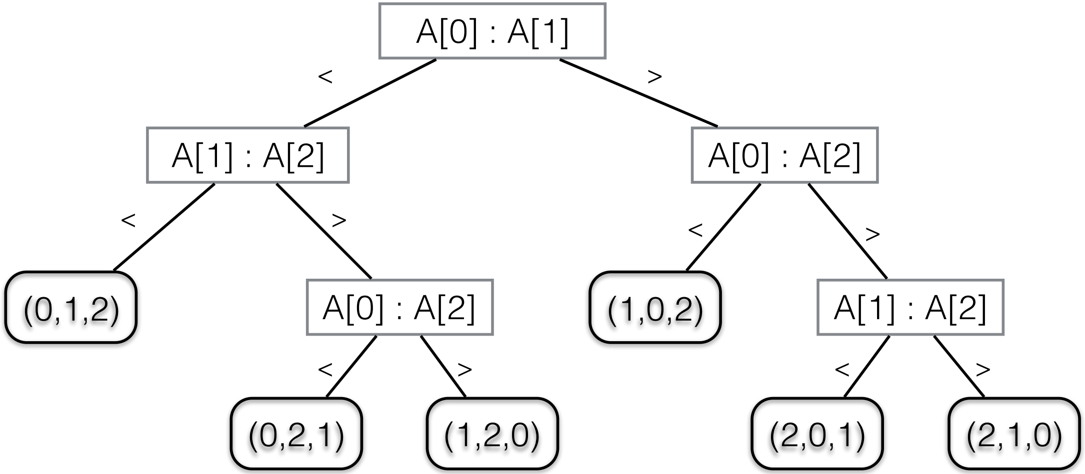
Example: $\boldsymbol{n} = \mathbf{3}$. Six possible permutations.

$A[0] < A[1]$

$A[1] > A[2]$



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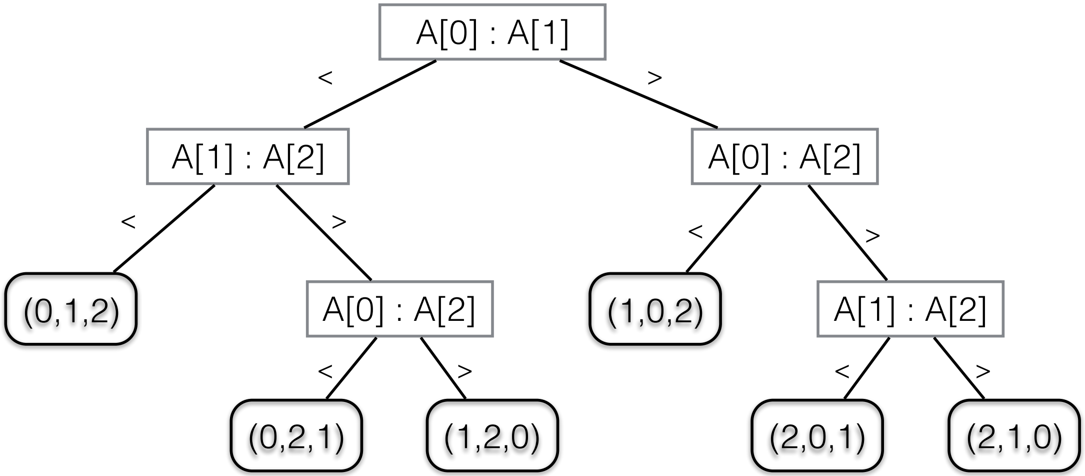

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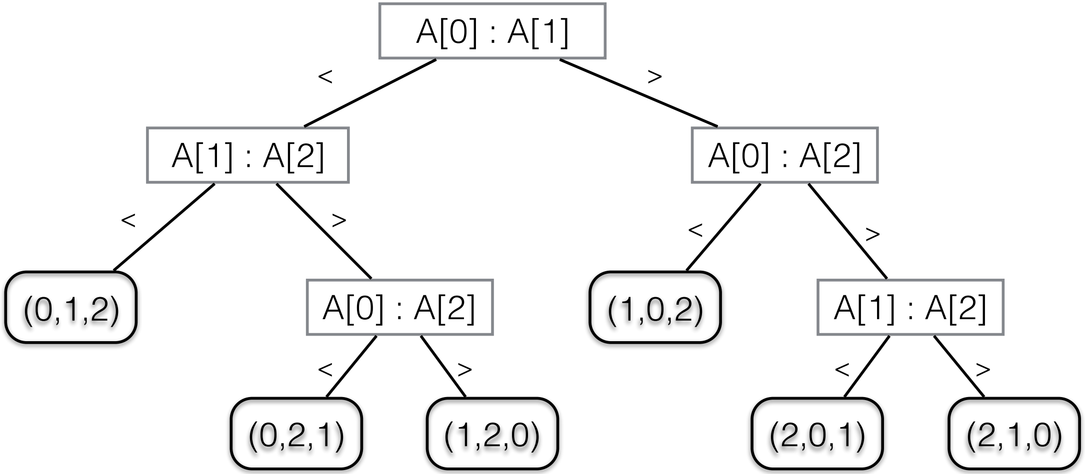
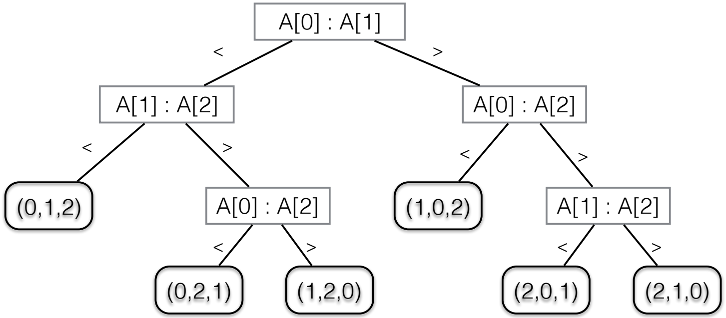


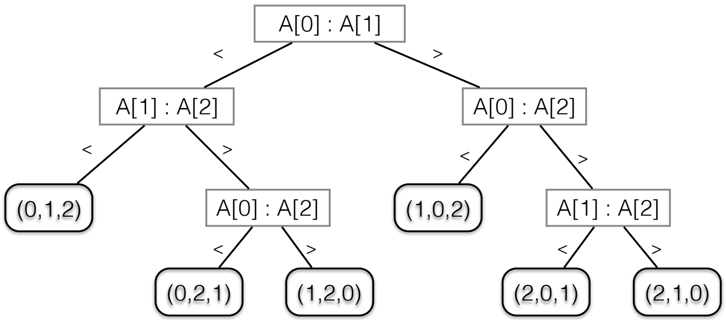
Max # comparisons: 3

Finishing Up



Scale to general n. Consider arbitrary decision tree.

Finishing Up



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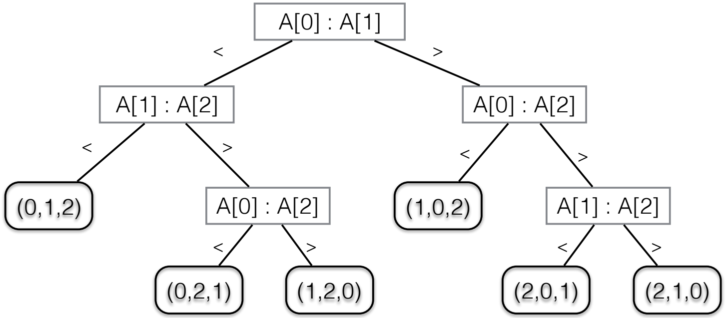

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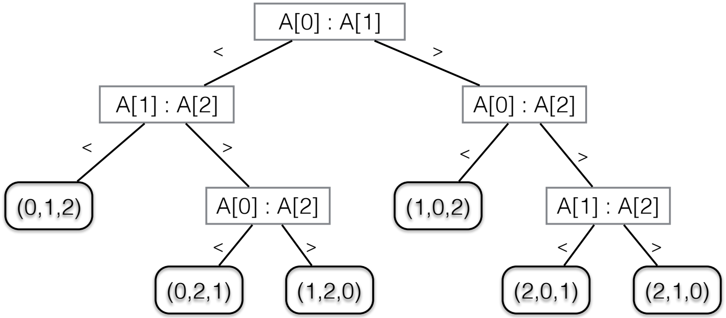

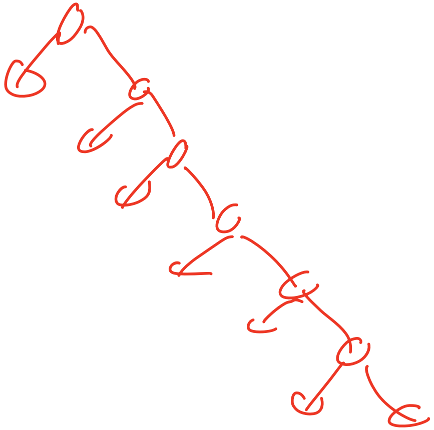

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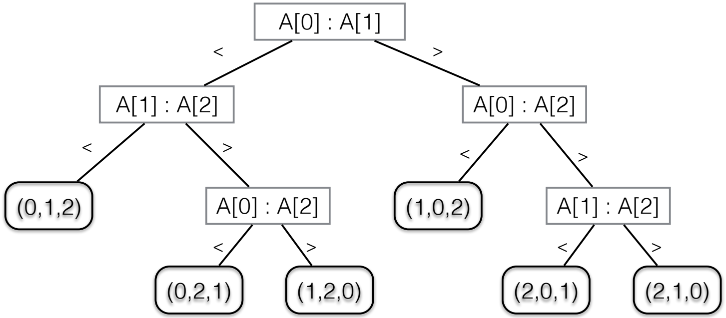

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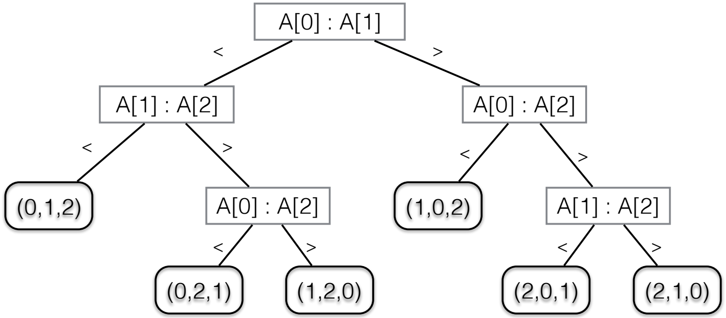

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$$\begin{aligned} \text{Max \# comparisons} &= \text{depth of tree} \\ &\geq \log_2(\#\text{ leaves}) \\ &= \log_2(n!) \\ &= \Theta(n \log n) \end{aligned}$$

Sorting Lower Bound Summary

Theorem

Every sorting algorithm in the comparison model must make at least $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

$\Omega(n \log n)$

Proof Sketch.

1. Lower bound on finding permutation $\pi \implies$ lower bound on sorting
2. Any algorithm for finding π is a binary decision tree with $n!$ leaves.
3. Any binary decision tree with $n!$ leaves has depth $\geq \log(n!) = \Theta(n \log n)$

$\implies$ Every algorithm has worst case number of comparisons at least $\Theta(n \log n)$. $\square$

“Linear-Time” Sorting

Bypassing the Lower Bound

What if we're *not* in the comparison model?

- Can do more than just compare elements.

Main example: *integers*.

- What is the 3rd bit of $A[0]$?
- Is $A[0] \ll k$ larger than $A[1] \gg c$?
- Is $A[0]$ even?

Same ideas apply to letters, strings, etc.

Counting Sort

Suppose $\boldsymbol{A}$ consists of $\boldsymbol{n}$ integers, all in $\{\mathbf{0}, \mathbf{1}, \ldots, \boldsymbol{k} - \mathbf{1}\}$.

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- Maintain an array $\boldsymbol{B}$ of length $\boldsymbol{k}$ initialized to all $\mathbf{0}$ $O(k)$
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- Scan through $\boldsymbol{B}$, output $\boldsymbol{i}$ exactly $\boldsymbol{B}[\boldsymbol{i}]$ times. $O(k)$

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Running time: $\boldsymbol{O}(\boldsymbol{n} + \boldsymbol{k})$

Bucket Sort: Counting Sort++

Often want to sort *objects* based on *keys*:

- Each object has a key: integer in $\{0, 1, \ldots, k-1\}$
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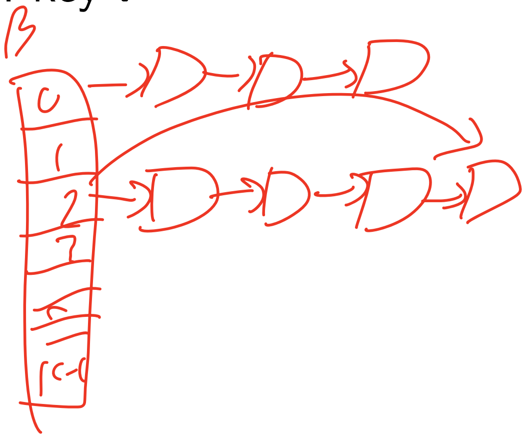

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Running time: $O(n + k)$

Stable: if two objects have same key, order between them after sorting is same as before.

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- Assume all numbers have exactly $\boldsymbol{d}$ digits (for simplicity)

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If you were sorting cards, with a number on each card, what might you do?

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Divide into $\mathbf{10}$ buckets by first digit, recurse on each bucket by second-digit, etc.

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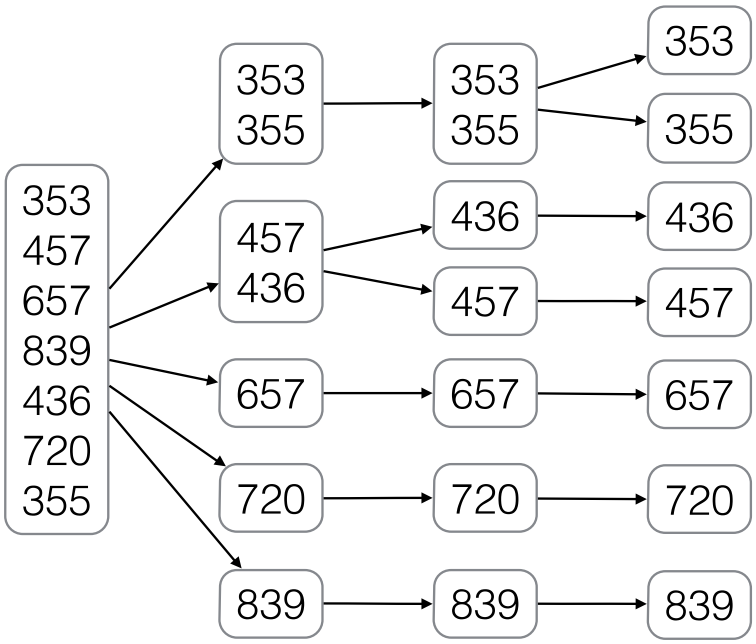

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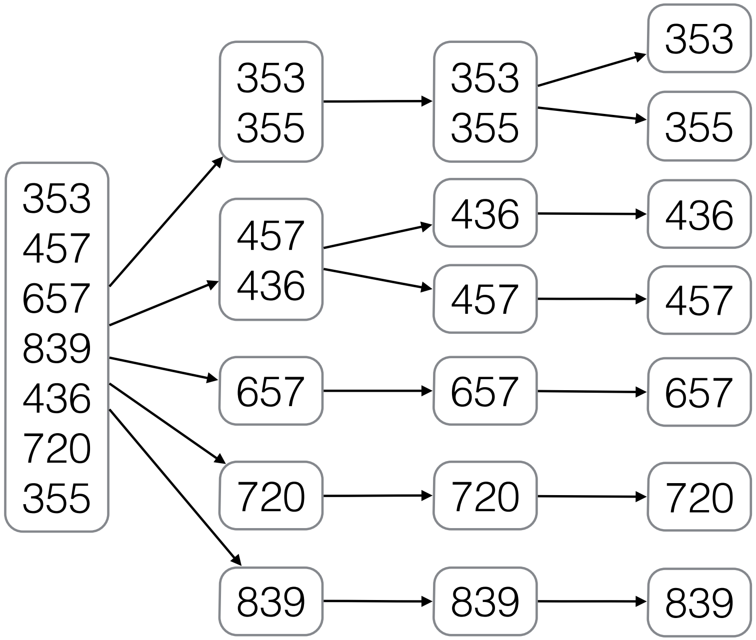

Works, but clunky

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Theorem

Radix sort from least significant to most significant is correct if the sort used on each digit is stable.

Least-Significant Radix Sort: Correctness

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Claim: After i'th iteration, correctly sorted by last i digits (interpreted as # in $[0, 10^i - 1]$).

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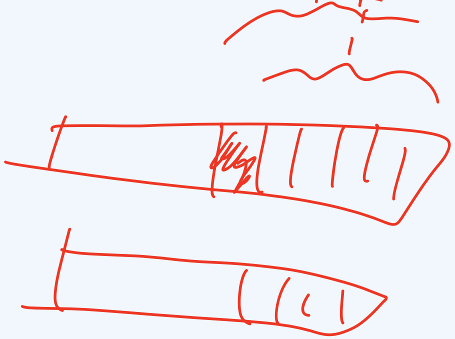

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Induction:

- Suppose correct for i
- After $i + 1$ sort:
  - If two numbers have different $i + 1$ digits, now correct.
  - If two number have same $i + 1$ digit, were correct and still correct by stability.

$\square$

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2 7 3

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Improve to $\boldsymbol{O(n)}$?

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Example: sorting integers between 0 and n^{10}. Then d should be about $\log_{10} n^{10} = 10\log_{10} n$, as required.