Lecture 6: Sorting Lower Bound and “Linear-Time” Sorting

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601.433/633 Introduction to Algorithms
Slides by Michael Dinitz

Introduction

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No: every algorithm in the comparison model must have worst-case running time $\Omega(n \log n)$.

Yes: If we assume extra structure for the elements, can do sorting in $O(n)$ time*

Sorting Lower Bound

Statement

Theorem

Any sorting algorithm in the comparison model must make at least $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

Lower bound on the number of comparisons – running time could be even worse!
Allows algorithm to reorder elements, copy them, move them, etc. for free.

$n! = n \cdot n-1 \cdots 2 \cdot 1$

$\log(n!) = \sum_{i=0}^{n-1} \log(n-i)$

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Why is this hard?

- Lower bound needs to hold for *all* algorithms
- How can we simultaneously reason about algorithms as different as mergesort, quicksort, heapsort, ...?

Sorting as Permutations

Think of an array $\boldsymbol{A}$ as a *permutation*: $\boldsymbol{A}[\boldsymbol{i}]$ is the $\pi(\boldsymbol{i})$'th smallest element

$$\boldsymbol{A} = [23, 14, 2, 5, 76]$$

$A' = [2, 5, 14, 23, 76]$

Corresponds to $\pi = (3, 2, 0, 1, 4)$:

$\pi(0) = 3$ $\quad$ $\pi(1) = 2$ $\quad$ $\pi(2) = 0$ $\quad$ $\pi(3) = 1$ $\quad$ $\pi(4) = 4$

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Lemma

Given $\boldsymbol{A}$ with $|\boldsymbol{A}| = \boldsymbol{n}$, if can sort in $\boldsymbol{T}(\boldsymbol{n})$ comparisons then can find π in $\boldsymbol{T}(\boldsymbol{n})$ comparisons

Sorting As Permutations (cont'd)

Lemma

Given A with $|A| = n$, if can sort in $T(n)$ comparisons then can find π in $T(n)$ comparisons

Proof Sketch.

- "Tag" each element of A with index:
  $[23, 14, 2, 5, 76] \to [(23, 0), (14, 1), (2, 2), (5, 3), (76, 4)]$
- Sort tagged A into tagged B with $T(n)$ comparisons:
  $[(2, 2), (5, 3), (14, 1), (23, 0), (76, 4)]$
- Iterate through to get π: $\pi(2) = 0, \pi(3) = 1, \pi(1) = 2, \pi(0) = 3, \pi(4) = 4$ □

Sorting As Permutations (cont'd)

Lemma

Given $\boldsymbol{A}$ with $|\boldsymbol{A}| = \boldsymbol{n}$, if can sort in $\boldsymbol{T(n)}$ comparisons then can find $\boldsymbol{\pi}$ in $\boldsymbol{T(n)}$ comparisons

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- Iterate through to get $\boldsymbol{\pi}$: $\boldsymbol{\pi}(\mathbf{2}) = \mathbf{0}, \boldsymbol{\pi}(\mathbf{3}) = \mathbf{1}, \boldsymbol{\pi}(\mathbf{1}) = \mathbf{2}, \boldsymbol{\pi}(\mathbf{0}) = \mathbf{3}, \boldsymbol{\pi}(\mathbf{4}) = \mathbf{4}$ $\square$

Corollary

If need at least $\boldsymbol{T(n)}$ comparisons to find $\boldsymbol{\pi}$, need at least $\boldsymbol{T(n)}$ comparisons to sort!

Generic Algorithm

Want to show that it takes $\boldsymbol{\Omega(n \log n)}$ comparisons to find $\boldsymbol{\pi}$ in comparison model.

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Arbitrary algorithm:

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- Rules out some possible permutations!
  - If $\boldsymbol{A[0] < A[1]}$ then $\boldsymbol{\pi(0) < \pi(1)}$
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- Continue until only one possible permutation.

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Remind you of anything?

Decision Trees

Model any algorithm as a *binary decision tree*

- Internal nodes: comparisons
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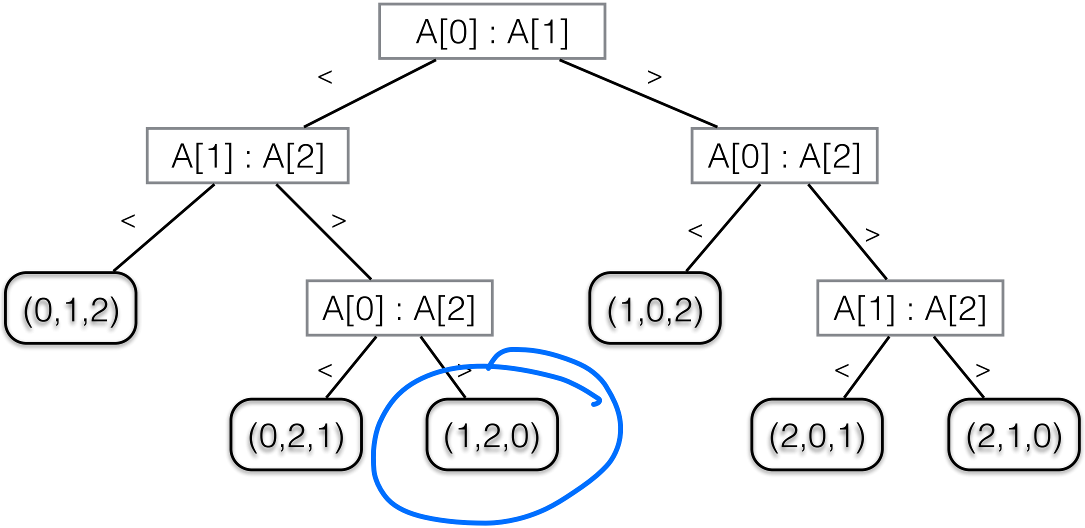
Example: $\boldsymbol{n} = \mathbf{3}$. Six possible permutations.



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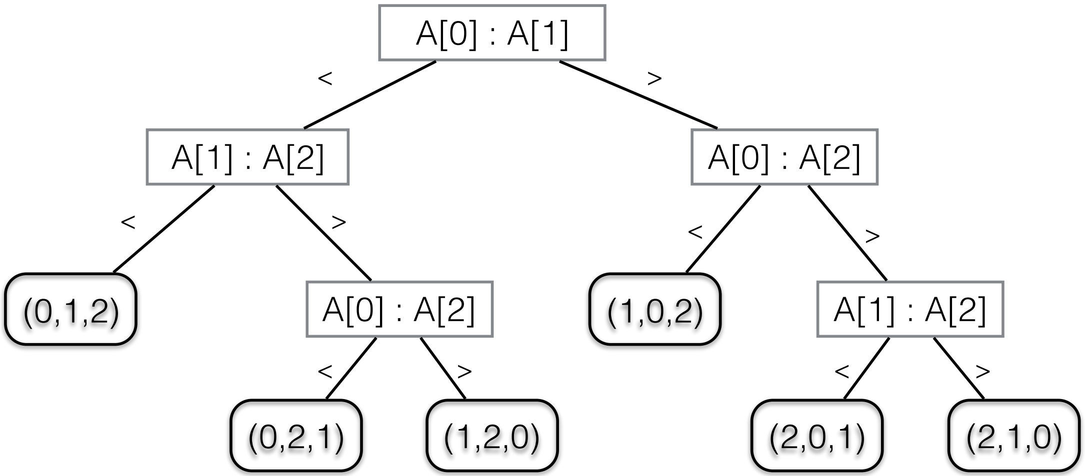

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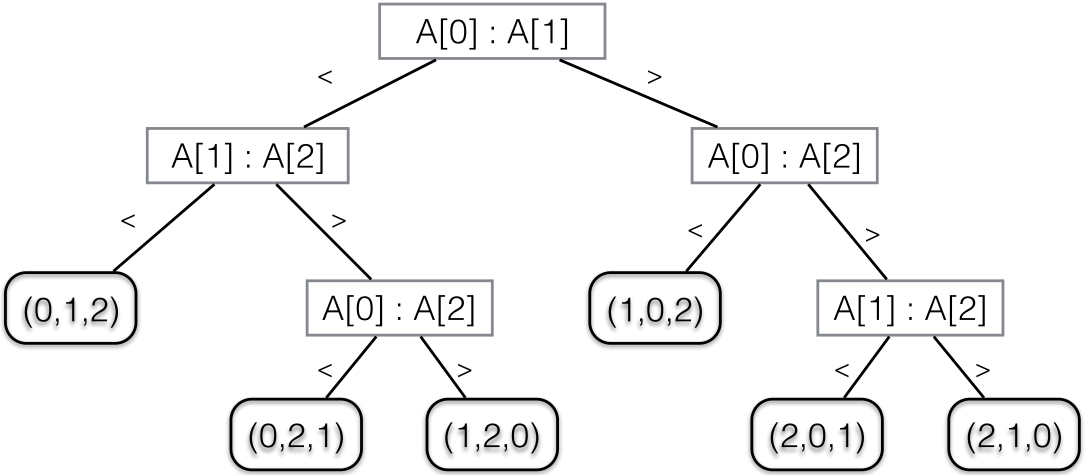
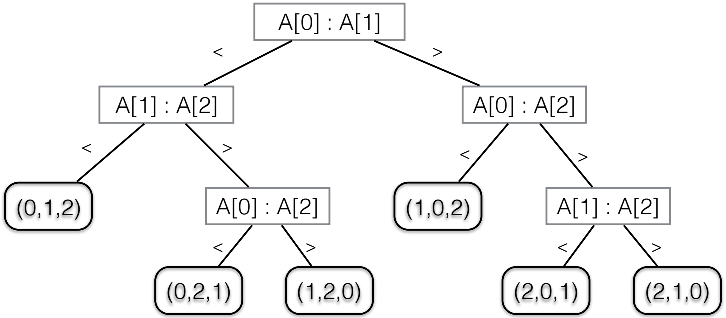


Max # comparisons: **3**

Finishing Up



Scale to general n. Consider arbitrary decision tree.

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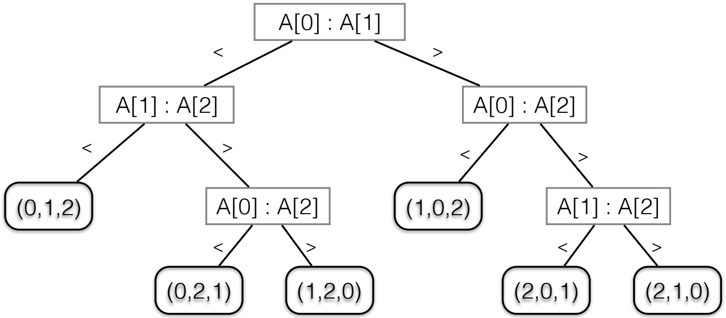

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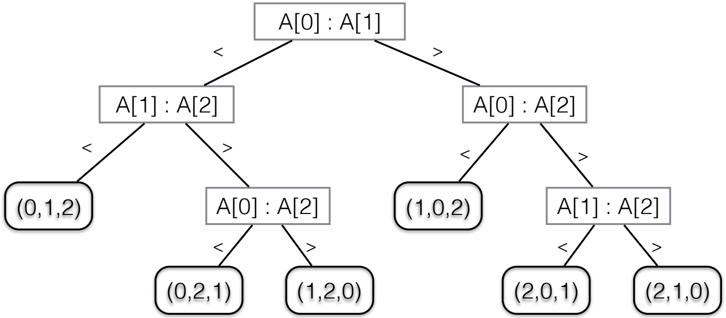

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Max # comparisons = depth of tree

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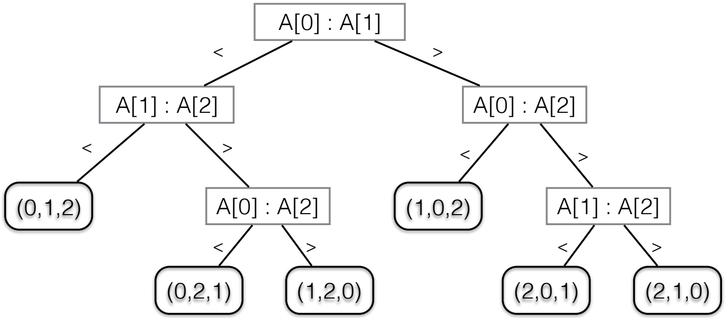

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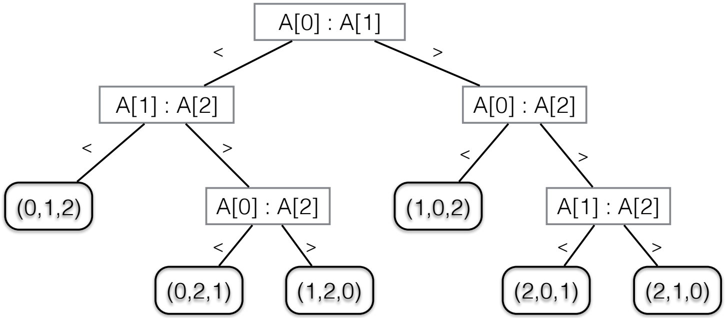

Scale to general $\boldsymbol{n}$. Consider arbitrary decision tree.

$$\begin{aligned}\text{Max \# comparisons} &= \text{depth of tree} \\ &\geq \mathbf{log_2}(\#\text{ leaves}) \\ &= \mathbf{log_2}(\boldsymbol{n}!)\end{aligned}$$

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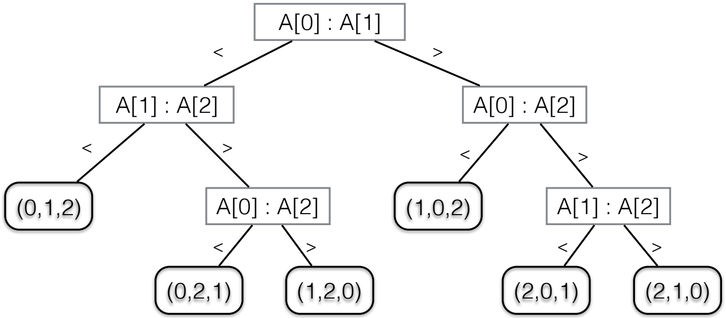

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$$\begin{aligned}\text{Max \# comparisons} &= \text{depth of tree}\\ &\geq \log_2(\#\text{ leaves})\\ &= \log_2(n!)\\ &= \Theta(n\log n)\end{aligned}$$

Sorting Lower Bound Summary

Theorem

Every sorting algorithm in the comparison model must make at least $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

Proof Sketch.

1. Lower bound on finding permutation $\pi \implies$ lower bound on sorting
2. Any algorithm for finding π is a binary decision tree with $n!$ leaves.
3. Any binary decision tree with $n!$ leaves has depth $\geq \log(n!) = \Theta(n \log n)$

$\implies$ Every algorithm has worst case number of comparisons at least $\Theta(n \log n)$. $\square$

“Linear-Time” Sorting

Bypassing the Lower Bound

What if we're *not* in the comparison model?

- Can do more than just compare elements.

Main example: *integers*.

- What is the $\mathbf{3}$rd bit of $\boldsymbol{A}[\mathbf{0}]$?
- Is $\boldsymbol{A}[\mathbf{0}]$ even?

Same ideas apply to letters, strings, etc.

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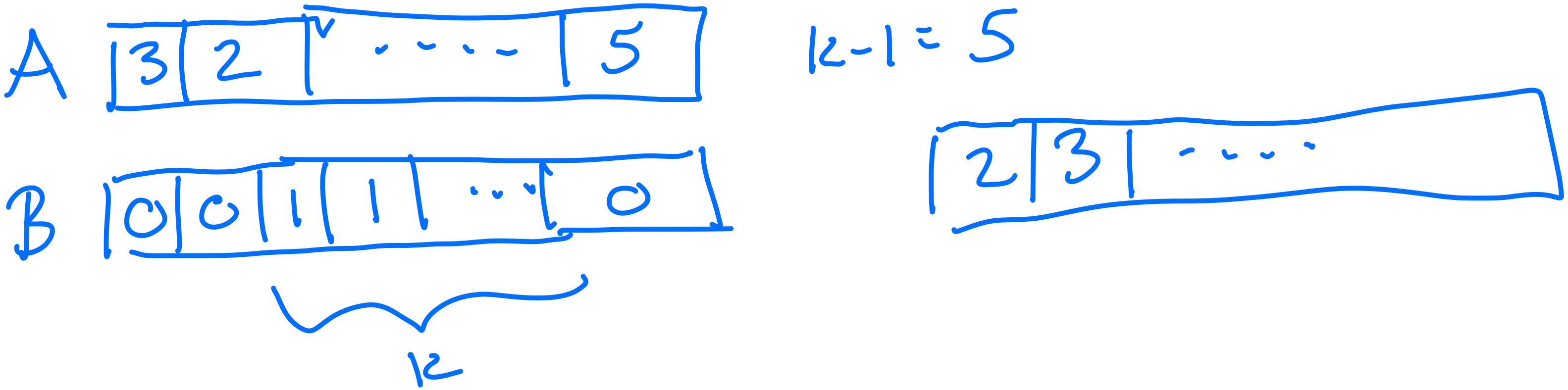

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Often want to sort *objects* based on *keys*:

- Each object has a key: integer in $\{0, 1, \ldots, k-1\}$
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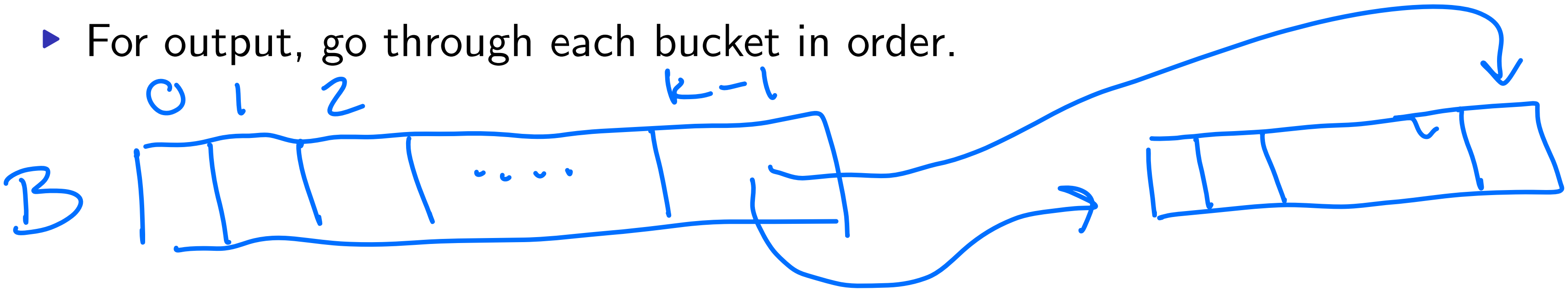

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Stable: if two objects have same key, order between them after sorting is same as before.

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If you were sorting cards, with a number on each card, what might you do?

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Divide into $\mathbf{10}$ buckets by first digit, recurse on each bucket by second-digit, etc.

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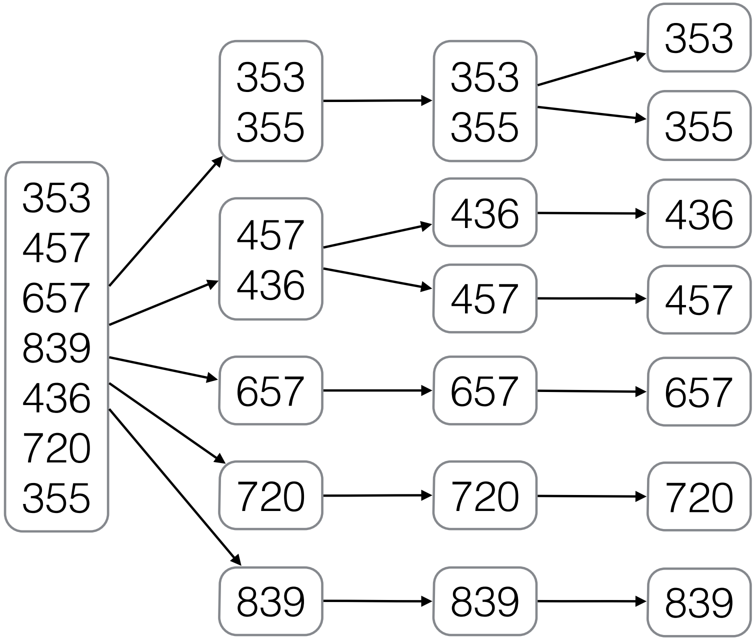

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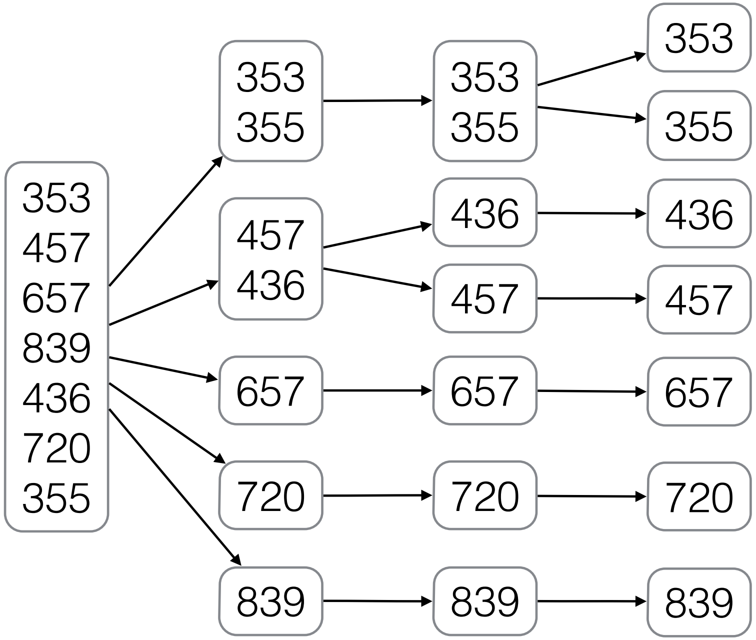

Works, but clunky

Radix-Sort: Algorithm (II)

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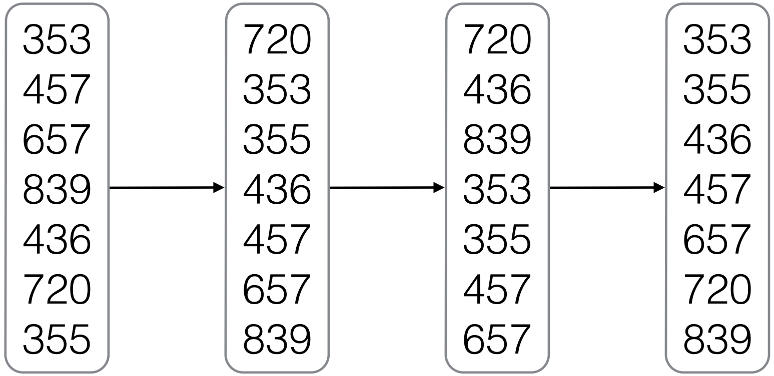

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353	720	720	353
457	353	436	355
657	355	839	436
839	436	353	457
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For iteration $\boldsymbol{i}$, use bucket sort where key is $\boldsymbol{i}$'th digit and object is number.

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For iteration i, use bucket sort where key is i'th digit and object is number.

Theorem

Radix sort from least significant to most significant is correct if the sort used on each digit is stable.

Least-Significant Radix Sort: Correctness

Proof.

Claim: After i'th iteration, correctly sorted by last i digits (interpreted as # in $[0, 10^i - 1]$).

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Proof.

Claim: After $\boldsymbol{i}$'th iteration, correctly sorted by last $\boldsymbol{i}$ digits (interpreted as # in $[\mathbf{0}, \mathbf{10}^{\boldsymbol{i}} - \mathbf{1}]$).
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Base case: After first iteration, correctly sorted by last digit

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- Suppose correct for i
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  - If two numbers have different $i + 1$ digits, now correct.
  - If two number have same $i + 1$ digit, were correct and still correct by stability.

□

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Time per bucket sort: $\boldsymbol{O(n+k) = O(n+10) = O(n)}$.

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Recall have n numbers, all numbers have d digits.

\# bucket sorts: d
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Improve to $O(n)$?

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- Kind of cheating: look at $\boldsymbol{b}$ digits in constant time.
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Set $b = \log_{10} n$. If $d = O(\log n)$, then time

$$O\left(\frac{d}{\log_{10} n}(n+n)\right) = O(n)$$

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Example: sorting integers between 0 and n^{10}. Then d should be about $\log_{10} n^{10} = 10\log_{10} n$, as required.