Lecture 6: Sorting Lower Bound and “Linear-Time” Sorting

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September 11, 2025
601.433/633 Introduction to Algorithms

Introduction

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- All algorithms we've seen so far have been in this model

No: every algorithm in the comparison model must have worst-case running time $\Omega(n \log n)$.

Yes: If we assume extra structure for the elements, can do sorting in $O(n)$ time*

Sorting Lower Bound

Statement

Theorem

Any sorting algorithm in the comparison model must make at least $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

Lower bound on the number of comparisons – running time could be even worse!
Allows algorithm to reorder elements, copy them, move them, etc. for free.

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Why is this hard?

- Lower bound needs to hold for *all* algorithms
- How can we simultaneously reason about algorithms as different as mergesort, quicksort, heapsort, . . . ?

Sorting as Permutations

Think of an array $\boldsymbol{A}$ as a *permutation*: $\boldsymbol{A}[\boldsymbol{i}]$ is the $\boldsymbol{\pi}(\boldsymbol{i})$'th smallest element

$$\boldsymbol{A} = [\mathbf{23}, \mathbf{14}, \mathbf{2}, \mathbf{5}, \mathbf{76}]$$

Corresponds to $\boldsymbol{\pi} = (\mathbf{3}, \mathbf{2}, \mathbf{0}, \mathbf{1}, \mathbf{4})$:

$$\boldsymbol{\pi}(\mathbf{0}) = \mathbf{3} \qquad \boldsymbol{\pi}(\mathbf{1}) = \mathbf{2} \qquad \boldsymbol{\pi}(\mathbf{2}) = \mathbf{0} \qquad \boldsymbol{\pi}(\mathbf{3}) = \mathbf{1} \qquad \boldsymbol{\pi}(\mathbf{4}) = \mathbf{4}$$

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Lemma

Given A with $|A| = n$, if can sort in $T(n)$ comparisons then can find π in $T(n)$ comparisons

Sorting As Permutations (cont'd)

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Proof Sketch.

- "Tag" each element of A with index:
  $[23, 14, 2, 5, 76] \to [(23, 0), (14, 1), (2, 2), (5, 3), (76, 4)]$
- Sort tagged A into tagged B with $T(n)$ comparisons:
  $[(2, 2), (5, 3), (14, 1), (23, 0), (76, 4)]$
- Iterate through to get π: $\pi(2) = 0, \pi(3) = 1, \pi(1) = 2, \pi(0) = 3, \pi(4) = 4$ $\square$

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Given A *with* $|A| = n$, *if can sort in* $T(n)$ *comparisons then can find* π *in* $T(n)$ *comparisons*

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Corollary

If need at least $T(n)$ *comparisons to find* π, *need at least* $T(n)$ *comparisons to sort!*

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Want to show that it takes $\Omega(n \log n)$ comparisons to find π in comparison model.

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- Starts with some comparison (e.g., compares $A[0]$ to $A[1]$)
- Rules out some possible permutations!
  - If $A[0] < A[1]$ then $\pi(0) < \pi(1)$
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Remind you of anything?

Decision Trees

Model any algorithm as a *binary decision tree*

- Internal nodes: comparisons
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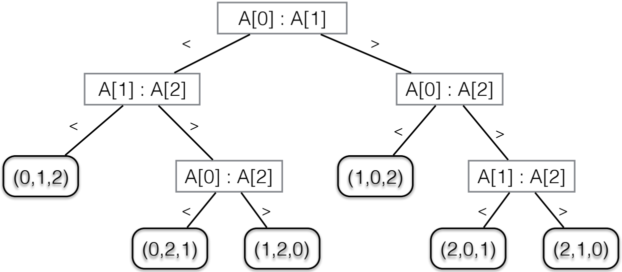
Example: $n = 3$. Six possible permutations.



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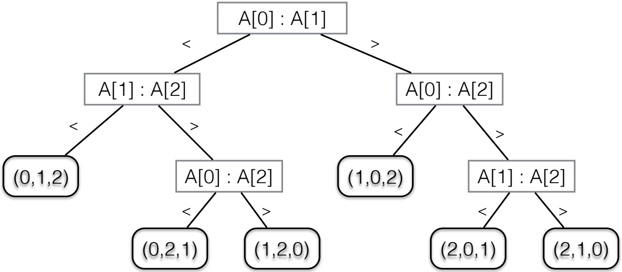

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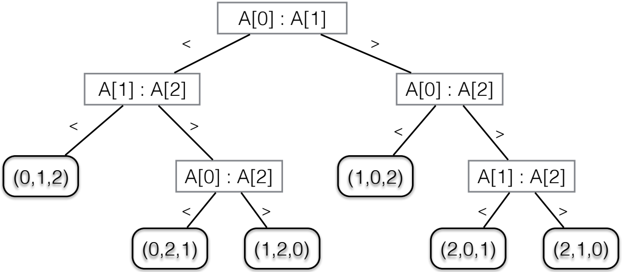
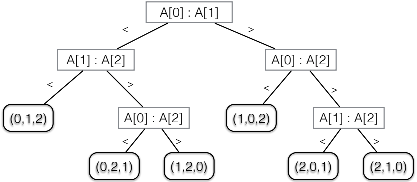


Max # comparisons: **3**

Finishing Up



Scale to general n. Consider arbitrary decision tree.

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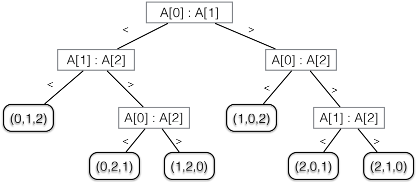

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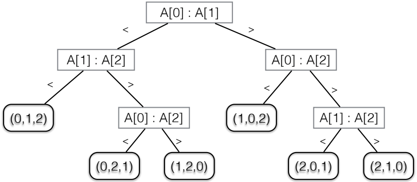

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Max # comparisons = depth of tree

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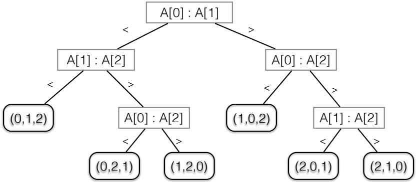

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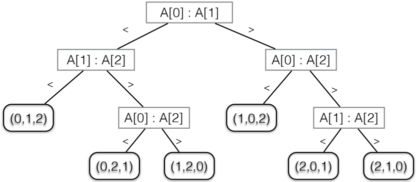

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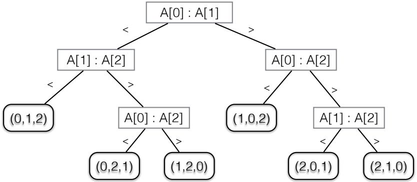

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$$\begin{aligned}
\text{Max \# comparisons} &= \text{depth of tree} \\
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&= \Theta(n \log n)
\end{aligned}$$

Sorting Lower Bound Summary

Theorem

Every sorting algorithm in the comparison model must make at least $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

Proof Sketch.

1. Lower bound on finding permutation $\pi \implies$ lower bound on sorting
2. Any algorithm for finding π is a binary decision tree with $n!$ leaves.
3. Any binary decision tree with $n!$ leaves has depth $\geq \log(n!) = \Theta(n \log n)$

$\implies$ Every algorithm has worst case number of comparisons at least $\Theta(n \log n)$. $\square$

“Linear-Time” Sorting

Bypassing the Lower Bound

What if we're *not* in the comparison model?

- Can do more than just compare elements.

Main example: *integers*.

- What is the $\mathbf{3}$rd bit of $A[0]$?
- Is $A[0] \ll k$ larger than $A[1] \gg c$?
- Is $A[0]$ even?

Same ideas apply to letters, strings, etc.

Counting Sort

Suppose $\boldsymbol{A}$ consists of $\boldsymbol{n}$ integers, all in $\{\mathbf{0}, \mathbf{1}, \ldots, \boldsymbol{k} - \mathbf{1}\}$.

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- Each object has a key: integer in $\{0, 1, \ldots, k-1\}$
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Stable: if two objects have same key, order between them after sorting is same as before.

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Setup:

- Numbers represented base 10 for historical reasons (all works fine in binary)
- Assume all numbers have exactly $\boldsymbol{d}$ digits (for simplicity)

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If you were sorting cards, with a number on each card, what might you do?

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Divide into $\mathbf{10}$ buckets by first digit, recurse on each bucket by second-digit, etc.

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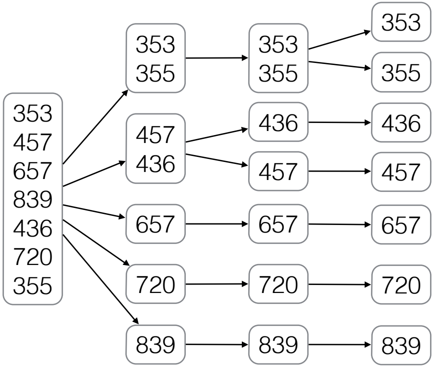

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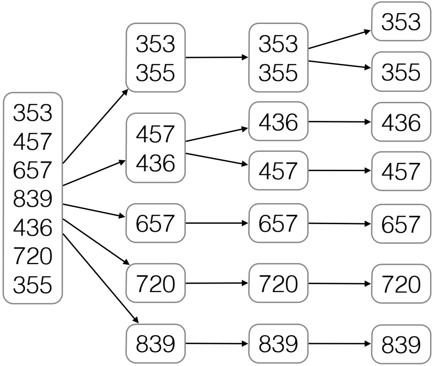

Works, but clunky

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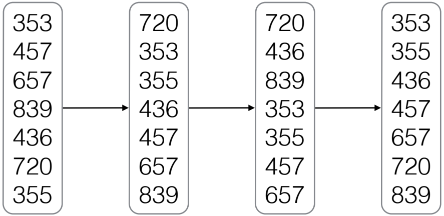

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	→		→		→	
353		720		720		353
457		353		436		355
657		355		839		436
839		436		353		457
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For iteration $\boldsymbol{i}$, use bucket sort where key is $\boldsymbol{i}$'th digit and object is number.

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For iteration i, use bucket sort where key is i'th digit and object is number.

Theorem

Radix sort from least significant to most significant is correct if the sort used on each digit is stable.

Least-Significant Radix Sort: Correctness

Proof.

Claim: After i'th iteration, correctly sorted by last i digits (interpreted as # in $[0, 10^i - 1]$).

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  - If two numbers have different $i + 1$ digits, now correct.
  - If two number have same $i + 1$ digit, were correct and still correct by stability.

□

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Improve to $O(n)$?

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Example: sorting integers between $\mathbf{0}$ and n^{10}. Then $\boldsymbol{d}$ should be about $\log_{10} n^{10} = 10\log_{10} n$, as required.