# Lecture 6: Sorting Lower Bound and "Linear-Time" Sorting

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601.433/633 Introduction to Algorithms
Slides by Michael Dinitz

# Introduction

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No: every algorithm in the comparison model must have worst-case running time $\Omega(n \log n)$.

Yes: If we assume extra structure for the elements, can do sorting in $O(n)$ time*

# Sorting Lower Bound

# Statement

**Theorem**

*Any sorting algorithm in the comparison model must make at least* $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

Lower bound on the number of comparisons – running time could be even worse!

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Why is this hard?

- Lower bound needs to hold for *all* algorithms
- How can we simultaneously reason about algorithms as different as mergesort, quicksort, heapsort, ...?

# Sorting as Permutations

Think of an array $A$ as a *permutation*: $A[i]$ is the $\pi(i)$'th smallest element

$$A = [23, 14, 2, 5, 76]$$

Corresponds to $\pi = (3, 2, 0, 1, 4)$:

$$\pi(0) = 3 \qquad \pi(1) = 2 \qquad \pi(2) = 0 \qquad \pi(3) = 1 \qquad \pi(4) = 4$$

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**Lemma**

*Given $A$ with $|A| = n$, if can sort in $T(n)$ comparisons then can find $\pi$ in $T(n)$ comparisons*

# Sorting As Permutations (cont'd)

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## Proof Sketch.

- "Tag" each element of $A$ with index:
  $[23, 14, 2, 5, 76] \to [(23, 0), (14, 1), (2, 2), (5, 3), (76, 4)]$
- Sort tagged $A$ into tagged $B$ with $T(n)$ comparisons:
  $[(2, 2), (5, 3), (14, 1), (23, 0), (76, 4)]$
- Iterate through to get $\pi$: $\pi(2) = 0, \pi(3) = 1, \pi(1) = 2, \pi(0) = 3, \pi(4) = 4$ □

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## Corollary

*If need at least $T(n)$ comparisons to find $\pi$, need at least $T(n)$ comparisons to sort!*

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Want to show that it takes $\Omega(n \log n)$ comparisons to find $\pi$ in comparison model.

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- Rules out some possible permutations!
  - If $A[0] < A[1]$ then $\pi(0) < \pi(1)$
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Remind you of anything?

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Model any algorithm as a *binary decision tree*

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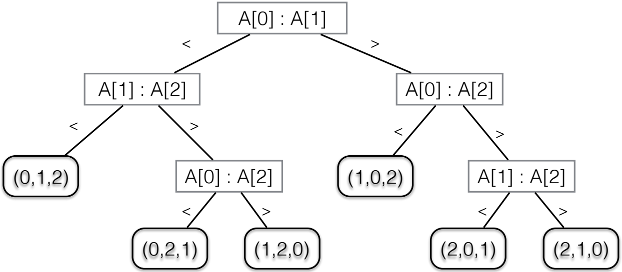
Example: $n = 3$. Six possible permutations.



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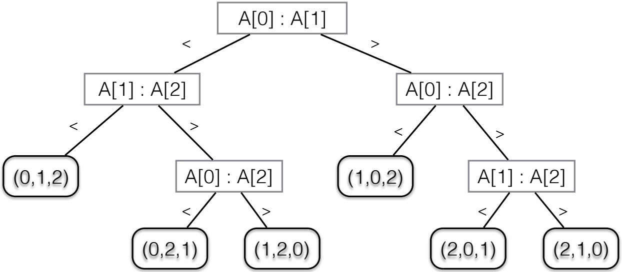


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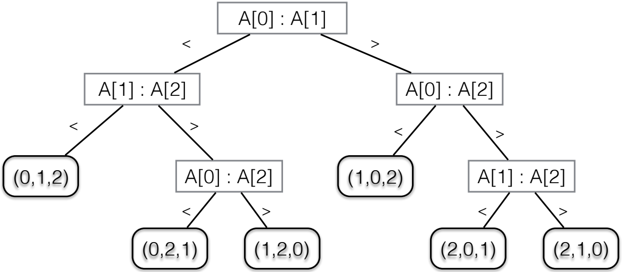
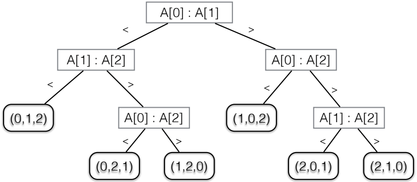



Max # comparisons: **3**

# Finishing Up




Scale to general $n$. Consider arbitrary decision tree.

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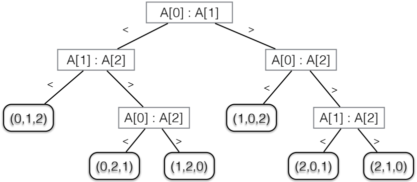


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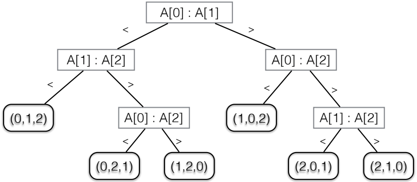


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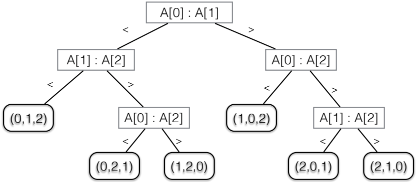


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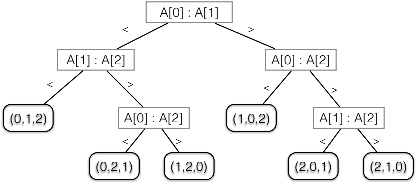


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\begin{aligned}
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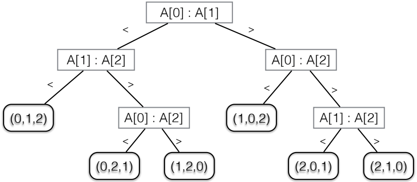


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# Sorting Lower Bound Summary

**Theorem**

*Every sorting algorithm in the comparison model must make at least* $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

**Proof Sketch.**

1. Lower bound on finding permutation $\pi \implies$ lower bound on sorting
2. Any algorithm for finding $\pi$ is a binary decision tree with $n!$ leaves.
3. Any binary decision tree with $n!$ leaves has depth $\geq \log(n!) = \Theta(n \log n)$

$\implies$ Every algorithm has worst case number of comparisons at least $\Theta(n \log n)$. □

# “Linear-Time” Sorting

# Bypassing the Lower Bound

What if we're *not* in the comparison model?

- Can do more than just compare elements.

Main example: *integers*.

- What is the $\mathbf{3}$rd bit of $\boldsymbol{A}[\mathbf{0}]$?
- Is $\boldsymbol{A}[\mathbf{0}]$ even?

Same ideas apply to letters, strings, etc.

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*Stable*: if two objects have same key, order between them after sorting is same as before.

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If you were sorting cards, with a number on each card, what might you do?

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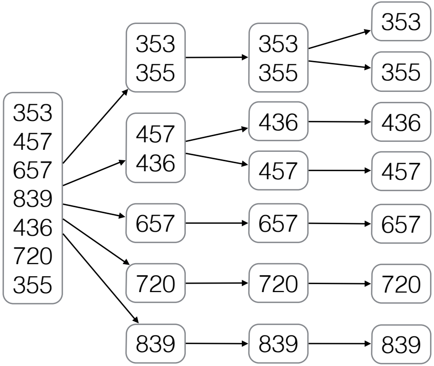

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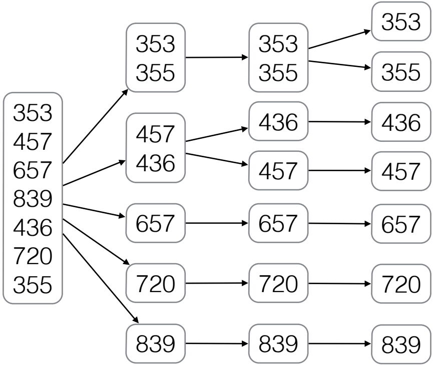


Works, but clunky

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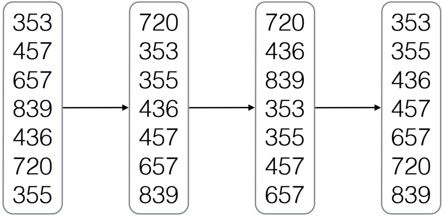

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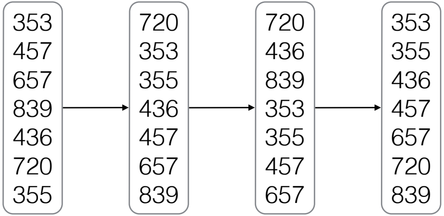


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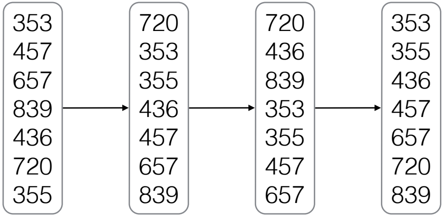


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**Theorem**

*Radix sort from least significant to most significant is correct if the sort used on each digit is stable.*

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- Suppose correct for $i$
- After $i + 1$ sort:
  - If two numbers have different $i + 1$ digits, now correct.
  - If two number have same $i + 1$ digit, were correct and still correct by stability.

□

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Improve to $O(n)$?

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Time per bucket sort: $O(n+k) = O(n+10^b)$
Total time: $O\left(\frac{d}{b}\left(n+10^b\right)\right)$

Set $b = \log_{10} n$. If $d = O(\log n)$, then time

$$O\left(\frac{d}{\log_{10} n}(n+n)\right) = O(n)$$

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Example: sorting integers between $0$ and $n^{10}$. Then $d$ should be about $\log_{10} n^{10} = 10 \log_{10} n$, as required.