Lecture 9: Priority Queues and Heaps

Michael Dinitz

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601.433/633 Introduction to Algorithms

Introduction

Priority Queues / Heaps: Like a queue/stack, but instead of FIFO/LIFO, by priority

- Insert(H, x): insert element x into heap H.
- Extract-Min(H): remove and return an element with smallest key
- Decrease-Key(H, x, k): decrease the key of x to k.
- Meld(H_1, H_2): replace heaps H_1 and H_2 with their union

Extra Operations:

- Find-Min(H): return the element with smallest key
- Delete(H, x): delete element x from heap H

Min-Heap, but can also do Max-Heap.

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Note: x is a *pointer* to an element. No way to lookup, so need a pointer to an element to change it.

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No! Sorting lower bound. But maybe can make one $O(1)$, other $O(\log n)$?

Today and State of the Art

State of the art: *strict Fibonacci Heaps*.

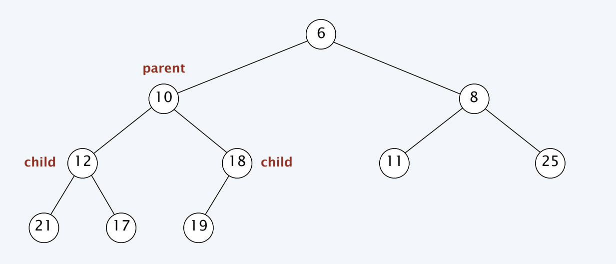
- Too complicated for this class, not practical. See CLRS 19 for Fibonacci Heaps.

Today: binary heaps (should be review), then binomial heaps

- Binomial heaps not quite as complicated as Fibonacci heaps, many of same core ideas

Binary Heaps

- Complete binary tree, except possibly at bottom level.
- Heap order: key of any node no larger than key of its children.



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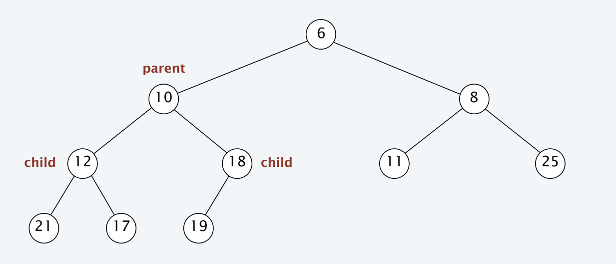


Properties:

- Since (almost) complete binary tree, depth $\mathbf{\Theta(\log n)}$
- Min must be at root

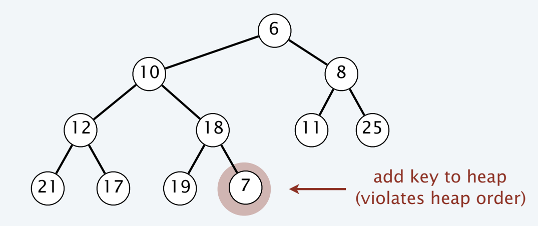
Representation:

- Pointers to root and rightmost leaf
- Every node has pointers to parent and children

Insert($\boldsymbol{H}, \boldsymbol{x}$)

Preserve heap *structure*: insert $\boldsymbol{x}$ into next open spot (bottom right, or left of new level if bottom level full)

- Might violate heap *order*!



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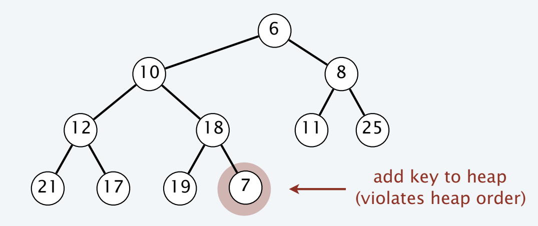
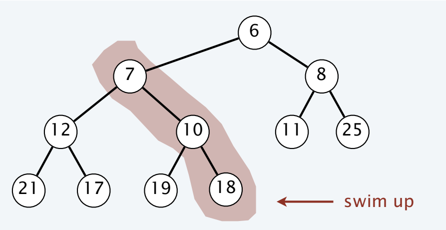


"Swim up": as long as $\boldsymbol{x}$ smaller than its parent, swap with parent



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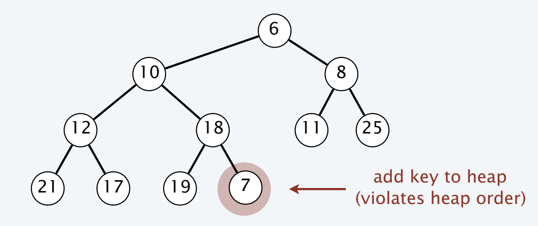

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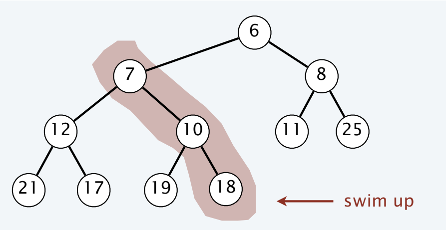

Running time: $\boldsymbol{O(\log n)}$ worst case (also amortized) via depth

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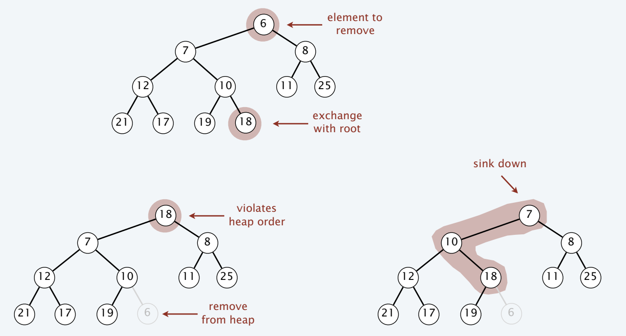

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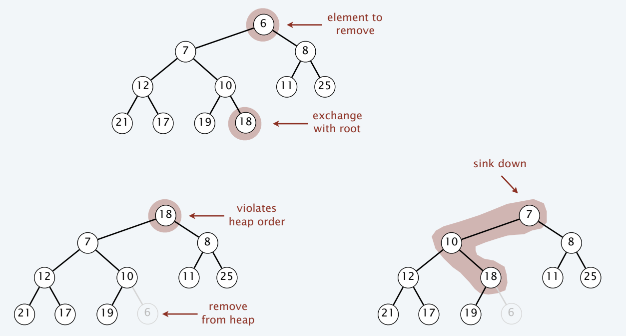

Running time: $O(\log n)$ worst case (via depth). Amortized: $O(1)$ (not obvious)

Decrease-Key(H, x, k)

Decrease key of x to k, "swim up" until heap order restored.

Running time: $O(\log n)$ (depth)

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$$\sum_{h=0}^{\log n} h\left(\frac{n}{2^h}\right) = n \sum_{h=0}^{\log n} \frac{h}{2^h} \leq O(n)$$

Amortized Extract-Min

Weights: $w(x)$ = depth of x

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Uses Inserts to "pay for" Extract-Mins.

Improvements

Downsides of binary heaps:

- Do at least as many Inserts as Extract-Mins! Want $O(1)$ Insert, $O(\log n)$ Extract-Min
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- Get Insert down to $O(1)$ (amortized)
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Fibonacci Heaps:

- Everything $O(1)$ (amortized) except $O(\log n)$ Extract-Min (amortized)

Binomial Heaps

Not based on binary tree anymore! Based on *binomial tree*.

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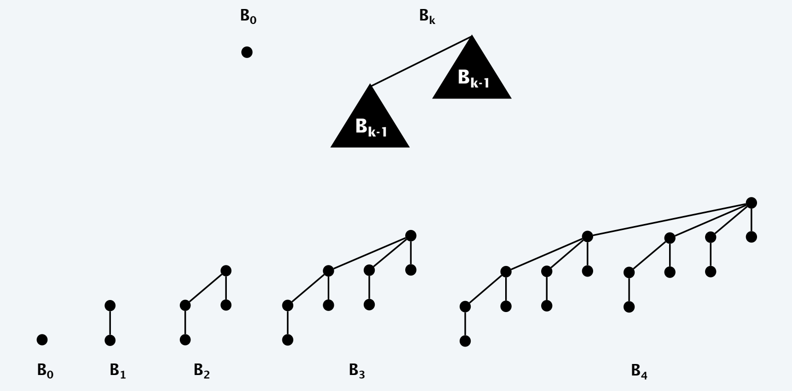
- B_0 = single node.
- B_k = one B_{k-1} linked to another B_{k-1}.

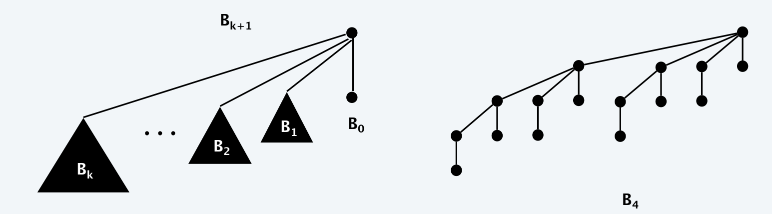


Structure Lemma

Lemma

The order $\boldsymbol{k}$ binomial tree $\boldsymbol{B_k}$ has the following properties:

1. *Its height is $\boldsymbol{k}$.*
2. *It has $\mathbf{2}^{\boldsymbol{k}}$ nodes*
3. *The degree of the root is $\boldsymbol{k}$*
4. *If we delete the root, we get $\boldsymbol{k}$ binomial trees $\boldsymbol{B_{k-1}, \ldots, B_0}$.*

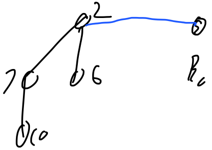


Binomial Heap

Definition

A *binomial heap* is a collection of binomial trees so that each tree is heap ordered, and there is exactly $\mathbf{0}$ or $\mathbf{1}$ tree of order $\boldsymbol{k}$ for each integer $\boldsymbol{k}$.

Keep roots of trees in linked list, from smallest order (not key!) to largest



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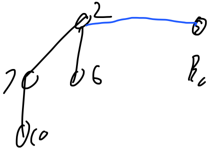

With n items, no choices about which binomial trees exist in heap!

- Write n in binary: $b_a b_{a-1} \ldots b_1 b_0$.
- Tree B_k exists if and only if $b_k = 1$

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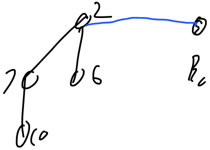

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- Write $\boldsymbol{n}$ in binary: $\boldsymbol{b_a b_{a-1} \dots b_1 b_0}$.
- Tree $\boldsymbol{B_k}$ exists if and only if $\boldsymbol{b_k = 1}$

$\Longrightarrow$ at most $\mathbf{log}\, \boldsymbol{n}$ trees, and by lemma each has height $\leq \mathbf{log}\, \boldsymbol{n}$

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Analyze all operations both worst-case and amortized.

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- Correct: each tree heap-ordered, so global min one of the roots
- Worst-case: $O(\log n)$
- Amortized: doesn't change potential, also $O(\log n)$.

Meld(H_1, H_2): Link

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Warmup: H_1, H_2 both single trees of same order k.

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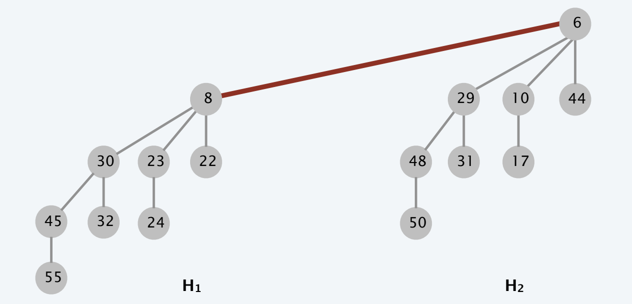

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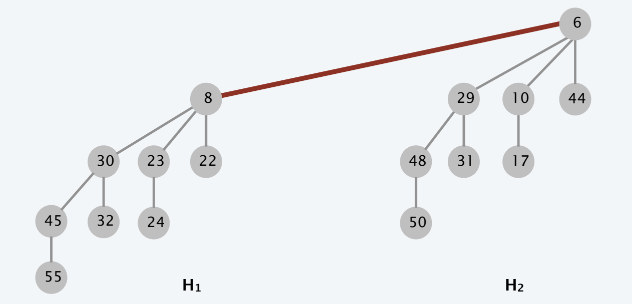
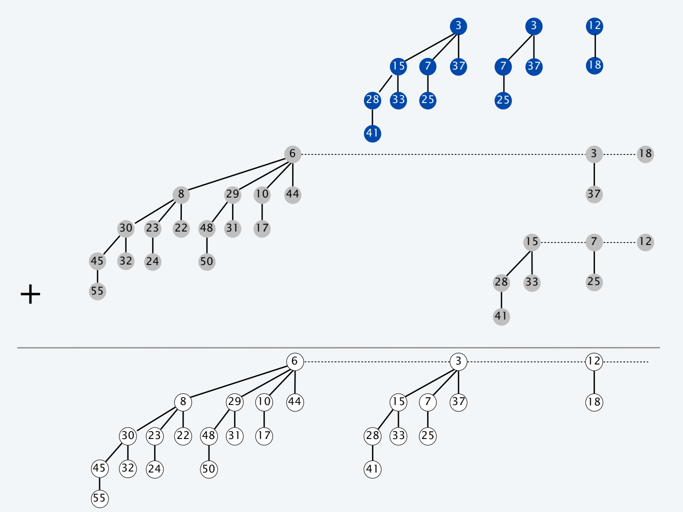


Link of two trees.

- Worst-case time: $O(1)$ (create a single link). Normalize: call 1
- $\Delta\Phi$: two trees to one: -1
- Amortized cost: $1 - 1 = 0 = O(1)$.

Meld(H_1, H_2): General Case

(Almost) just like binary addition!



Meld(H_1, H_2): Analysis

Easy to prove correct (exercise for home).

Running time:

- Worst case: $O(1)$ per "order" $k \implies \leq O(\log n)$
- Amortized: Potential does not go up, but could stay the same $\implies O(\log n)$ amortized

Insert(H, x)

Use Meld:

- Create new heap H' with one B_0 consisting of just x
- Meld(H, H')

Correctness: Obvious

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Running Time:

- Worst case: $O(\log n)$ (via Meld)
- Amortized:
  - Like incrementing a binary counter!
  - If we link k trees, potential goes down by $k - 1$
  - Cost = # links plus 1 (for making new heap)
  - Amortized cost = $k + 1 + \Delta\Phi = k + 1 - (k - 1) = 2 = O(1)$

Extract-Min(H)

Use Meld again!

- $O(\log n)$ to Find-Min: one of the roots.
- Delete and return this root: tree turns into a new heap!
- Meld with original heap (minus the tree)

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Running Time:

- Worst-Case: $O(\log n)$ from creating new heap, Meld
- Amortized:
  - Potential can go up! But by at most $\log n$
  - Amortized time at most $O(\log n) + \log n = O(\log n)$