Lecture 9: Priority Queues and Heaps

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601.433/633 Introduction to Algorithms
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Analysis

Do an increment, flips $\boldsymbol{k}$ bits $\implies$ true cost is $\boldsymbol{k}$.

- # $\mathbf{0}$'s flipped to $\mathbf{1}$: $\mathbf{1}$
- # $\mathbf{1}$'s flipped to $\mathbf{0}$: $\boldsymbol{k} - \mathbf{1}$

Flipping $\mathbf{1}$ to $\mathbf{0}$ paid for by bank! Costs $\mathbf{1}$, bank decreases by $\mathbf{1}$
$\implies$ amortized cost at most $\mathbf{1}$ (cost of flipping $\mathbf{0}$ to $\mathbf{1}$) plus $\mathbf{1}$ (increase in bank for that bit) $= \mathbf{2}$

Global: Change in *total* bank is $-(\boldsymbol{k} - \mathbf{1}) + \mathbf{1} = -\boldsymbol{k} + \mathbf{2}$
$\implies$ amortized cost $= \boldsymbol{c} + \boldsymbol{\Delta L} = \boldsymbol{k} + (-\boldsymbol{k} + 2) = 2$

Potential function: let $\boldsymbol{\Phi} = \#\mathbf{1}$'s in counter.
$\implies$ amortized cost $= \boldsymbol{c} + \boldsymbol{\Delta\Phi} = \boldsymbol{k} + (-\boldsymbol{k} + 2) = 2$

Introduction

Priority Queues / Heaps: Like a queue/stack, but instead of FIFO/LIFO, by priority

- Insert($\boldsymbol{H}, \boldsymbol{x}$): insert element $\boldsymbol{x}$ into heap $\boldsymbol{H}$.
- Extract-Min($\boldsymbol{H}$): remove and return an element with smallest key
- Decrease-Key($\boldsymbol{H}, \boldsymbol{x}, \boldsymbol{k}$): decrease the key of $\boldsymbol{x}$ to $\boldsymbol{k}$.
- Meld($\boldsymbol{H_1}, \boldsymbol{H_2}$): replace heaps $\boldsymbol{H_1}$ and $\boldsymbol{H_2}$ with their union

Extra Operations:

- Find-Min($\boldsymbol{H}$): return the element with smallest key
- Delete($\boldsymbol{H}, \boldsymbol{x}$): delete element $\boldsymbol{x}$ from heap $\boldsymbol{H}$

Min-Heap, but can also do Max-Heap.

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Min-Heap, but can also do Max-Heap.

Note: x is a *pointer* to an element. No way to lookup, so need a pointer to an element to change it.

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	Insert	Extract-Min	Decrease-Key	Meld
Linked List				

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Question: Can we make Insert and Extract-Min both $O(1)$, even amortized?

No! Sorting lower bound. But maybe can make one $O(1)$, other $O(\log n)$?

Today and State of the Art

State of the art: *strict Fibonacci Heaps*.

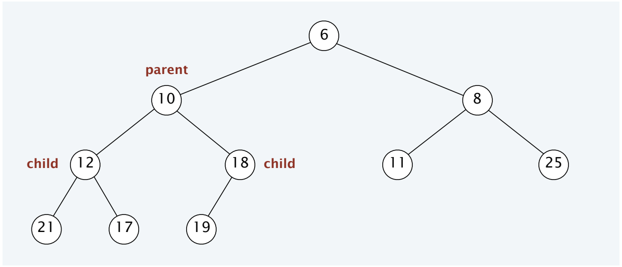
- Too complicated for this class, not practical. See CLRS 19 for Fibonacci Heaps.

Today: binary heaps (should be review), then binomial heaps

- Binomial heaps not quite as complicated as Fibonacci heaps, many of same core ideas

Binary Heaps

- Complete binary tree, except possibly at bottom level.
- Heap order: key of any node no larger than key of its children.



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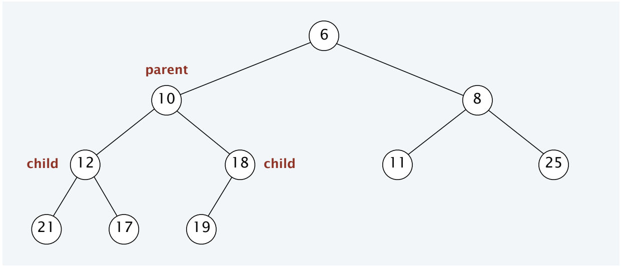


Properties:

- Since (almost) complete binary tree, depth $\Theta(\log n)$
- Min must be at root

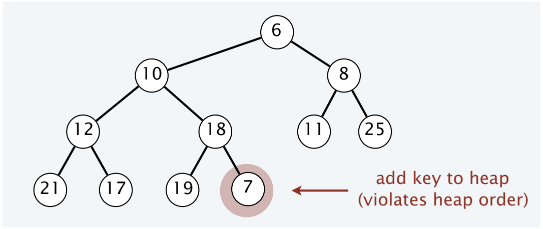
Representation:

- Pointers to root and rightmost leaf
- Every node has pointers to parent and children

Insert($\boldsymbol{H}, \boldsymbol{x}$)

Preserve heap *structure*: insert $\boldsymbol{x}$ into next open spot (bottom right, or left of new level if bottom level full)

- Might violate heap *order*!



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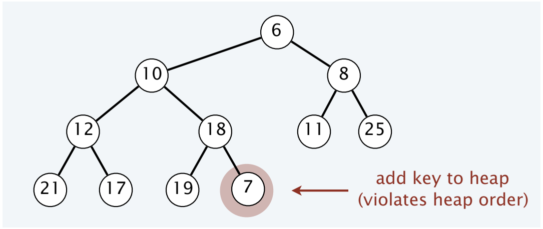
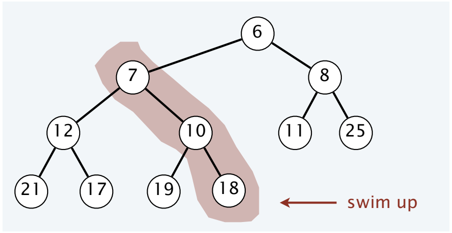


"Swim up": as long as $\boldsymbol{x}$ smaller than its parent, swap with parent



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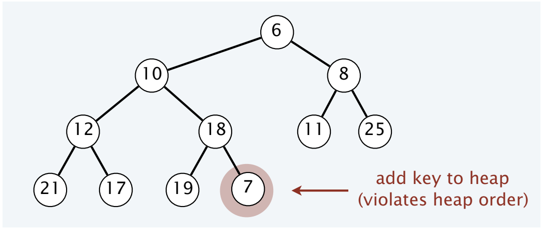

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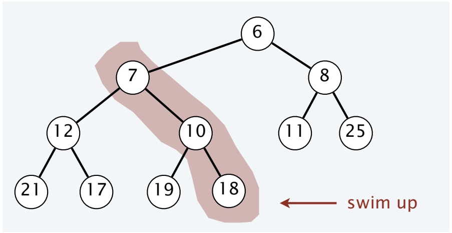

Running time: $\boldsymbol{O(\log n)}$ worst case (also amortized) via depth

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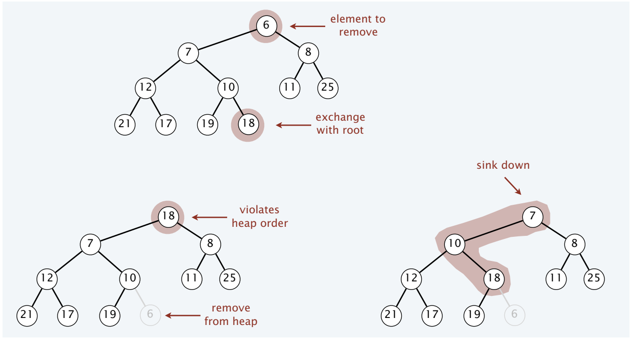

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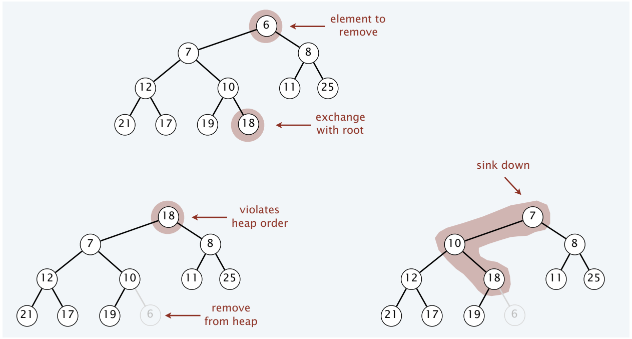

Running time: $\boldsymbol{O(\log n)}$ worst case (via depth). Amortized: $\boldsymbol{O(1)}$ (not obvious)

Decrease-Key($\boldsymbol{H, x, k}$)

Decrease key of $\boldsymbol{x}$ to $\boldsymbol{k}$, “swim up” until heap order restored.

Running time: $\boldsymbol{O(\log n)}$ (depth)

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    - Nodes at height h might have to sink down h.
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$$\sum_{h=0}^{\log n} h\left(\frac{n}{2^h}\right) = n \sum_{h=0}^{\log n} \frac{h}{2^h} \leq O(n)$$

Amortized Extract-Min

Weights: $w(x)$ = depth of x

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Extract-Min:

- True cost: height $h = \Theta(\log n)$ of tree, plus $O(1)$ (for initial swap).
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Uses Inserts to "pay for" Extract-Mins.

Improvements

Downsides of binary heaps:

- Do at least as many Inserts as Extract-Mins! Want $O(1)$ Insert, $O(\log n)$ Extract-Min
- Meld in $O(n)$ is better than trivial, but still not great.

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Binomial Heaps:

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Fibonacci Heaps:

- Everything $O(1)$ (amortized) except $O(\log n)$ Extract-Min (amortized)

Binomial Heaps

Not based on binary tree anymore! Based on *binomial tree*.

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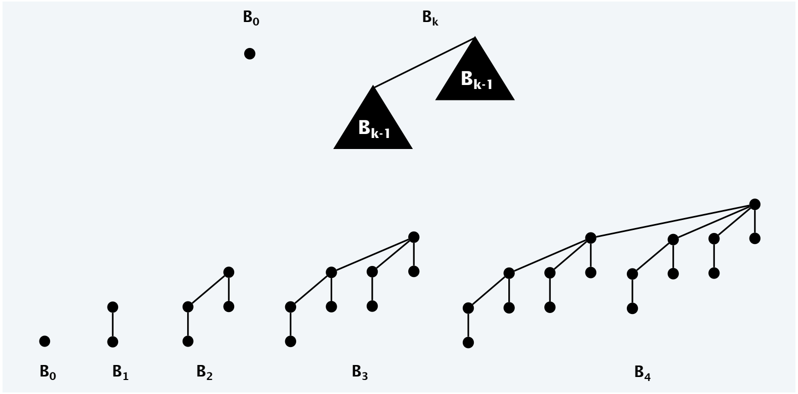
- B_0 = single node.
- B_k = one B_{k-1} linked to another B_{k-1}.

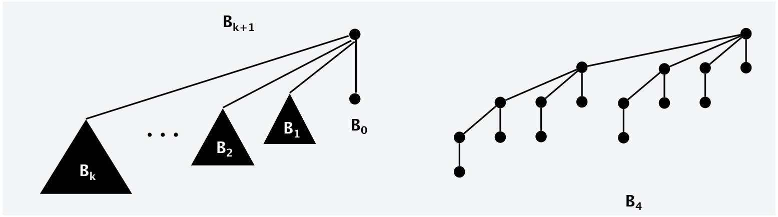


Structure Lemma

Lemma

The order k binomial tree B_k has the following properties:

1. *Its height is k.*
2. *It has 2^k nodes*
3. *The degree of the root is k*
4. *If we delete the root, we get k binomial trees $B_{k-1}, \ldots, B_0$.*

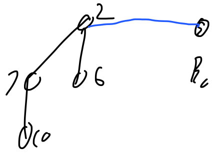


Binomial Heap

Definition

A *binomial heap* is a collection of binomial trees so that each tree is heap ordered, and there is exactly **0** or **1** tree of order ***k*** for each integer ***k***.

Keep roots of trees in linked list, from smallest order (not key!) to largest



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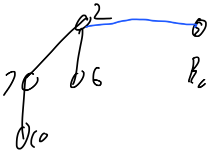

With $\boldsymbol{n}$ items, no choices about which binomial trees exist in heap!

- Write $\boldsymbol{n}$ in binary: $\boldsymbol{b_a b_{a-1} \dots b_1 b_0}$.
- Tree $\boldsymbol{B_k}$ exists if and only if $\boldsymbol{b_k = 1}$

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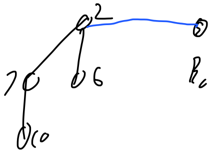

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$\implies$ at most $\boldsymbol{\log n}$ trees, and by lemma each has height $\leq \boldsymbol{\log n}$

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- Correct: each tree heap-ordered, so global min one of the roots
- Worst-case: $O(\log n)$
- Amortized: doesn't change potential, also $O(\log n)$.

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Warmup: $\boldsymbol{H_1}, \boldsymbol{H_2}$ both single trees of same order $\boldsymbol{k}$.

- Union has size $\mathbf{2^k + 2^k = 2^{k+1}}$: just a single $\boldsymbol{B_{k+1}}$
- Easy to make a $\boldsymbol{B_{k+1}}$ out of two $\boldsymbol{B_k}$'s!

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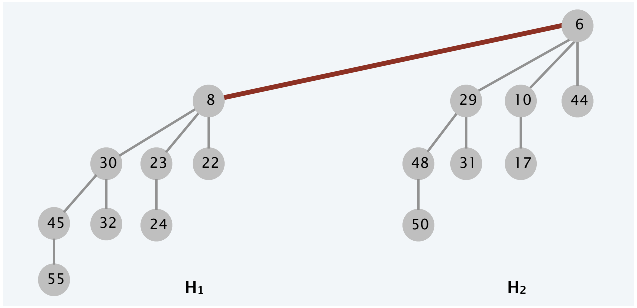

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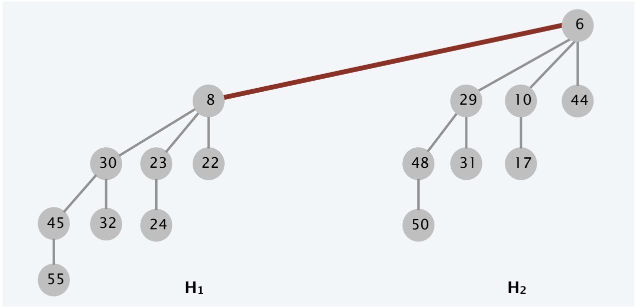
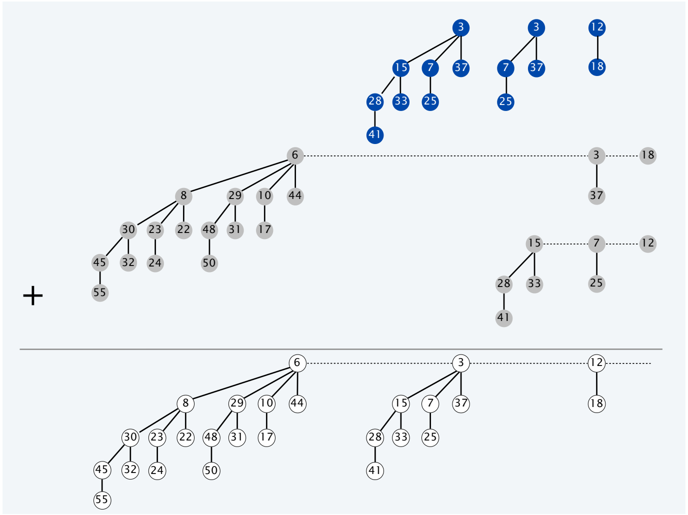


Link of two trees.

- Worst-case time: $O(1)$ (create a single link). Normalize: call $\mathbf{1}$
- $\Delta\Phi$: two trees to one: -1
- Amortized cost: $\mathbf{1 - 1 = 0} = O(1)$.

Meld(H_1, H_2): General Case

(Almost) just like binary addition!



Meld(H_1, H_2): Analysis

Easy to prove correct (exercise for home).

Running time:

- Worst case: $O(1)$ per "order" $k \implies \leq O(\log n)$
- Amortized: Potential does not go up, but could stay the same $\implies O(\log n)$ amortized

Insert($\boldsymbol{H}, \boldsymbol{x}$)

Use Meld:

- Create new heap $\boldsymbol{H'}$ with one $\boldsymbol{B_0}$ consisting of just $\boldsymbol{x}$
- Meld($\boldsymbol{H}, \boldsymbol{H'}$)

Correctness: Obvious

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Running Time:

- Worst case: $O(\log n)$ (via Meld)
- Amortized:
  - Like incrementing a binary counter!
  - If we link k trees, potential goes down by $k - 1$
  - Cost $=$ # links plus 1 (for making new heap)
  - Amortized cost $= k + 1 + \Delta\Phi = k + 1 - (k - 1) = 2 = O(1)$

Extract-Min($\boldsymbol{H}$)

Use Meld again!

- $\boldsymbol{O(\log n)}$ to Find-Min: one of the roots.
- Delete and return this root: tree turns into a new heap!
- Meld with original heap (minus the tree)

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Running Time:

- Worst-Case: $O(\log n)$ from creating new heap, Meld
- Amortized:
    - Potential can go up! But by at most $\log n$
    - Amortized time at most $O(\log n) + \log n = O(\log n)$